# DIMENSIONS IN ENGINEERING THEORY

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# PREFACE

THE attention given in elementary text-books on engineering theory to the important and interesting subject of dimensions varies considerably. In some books the subject is ignored entirely; in some it is dealt with briefly and superficially in an appendix; while in others, the treatment of the subject, although adequate, lacks coherence because the references to dimensions are incidental. For an orderly and connected account of the theory of dimensions, books on advanced mathematics or specialised monographs have to be consulted and works of this character are not very suitable for elementary students. There seems to be a real need for a small work, dealing entirely with the subject of dimensions and their applications to engineering theory, in an elementary way. This book has been written with the object of meeting this need.

The opening chapter deals with fundamental units, and here an attempt has been made to correct some misleading ideas about the unit of mass in so-called gravitational systems of units. The basic principles of dimensions are then explained in reference to dynamical quantities, and the applications of dimensional analysis to the calculation of conversion factors, the checking of formulae, and the investigation of the nature of physical laws are then dealt with and illustrated. Some difficulties in dimensional theory, usually treated rather lightly, are discussed fully, and an attempt has been made to show that the inconsistency of the identical dimensions of the dissimilar quantities, torque and work, can be avoided by the use of an auxiliary dimensional symbol in rotational dynamics. Reasons are also given for the view that a special dimensional symbol is best retained for temperature. In the concluding chapter, dealing with the dimensions of electrical quantities,

the difficulties which arise from the text-book definitions of permittivity and permeability have been explained.

It is hoped that this book will be acceptable to elementary students of all branches of engineering theory as an inexpensive supplement to their regular text-books. It is also hoped that the book will be found useful and interesting by practical engineers as a rudimentary introduction to a branch of applied mathematics which is of rapidly increasing importance in modern engineering theory.

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# CHAPTER I

## UNITS

Measures of Physical Quantities. Every quantitative statement about an objective magnitude is necessarily composed of two parts or factors; a number and a statement of the unit of measurement. The number is the mathematical ratio of the magnitude to that of the specified unit. Similarly, the ultimate end of all applied mathematics is the numerical evaluation, by the working of an arithmetical sum, of the magnitude of some physical quantity which is inferred from the known magnitudes of others. The answer to the sum will be intelligible only if the unit applicable to the calculated number is known with certainty. Thus, the number 4·2 may be part of the statement of a length, but this number is meaningless till it is known whether it refers to centimetres, feet, yards or miles, as the case may be. Similarly the answer to a calculation of an amount of energy may give a numerical answer 4·2, but this is unintelligible till it is known whether it refers to ergs, foot-lbs., or kilowatt hours.

Any physical quantity can be completely defined by a number and any arbitrarily chosen unit, provided that the unit is exactly specified. A collection of units for the measurement of physical quantities is known as a system of units, and, in such a system, the various units may be either arbitrarily defined, or they may be made to depend in a simple way on other units. Thus, the unit of volume, the cubic-foot, depends in a very simple way on another unit, the foot of length, but another volume unit, the gallon, has no such simple connection with any kind of length unit. An absolute system of units is one in which the arbitrarily defined units are the fewest possible, and in which the other

units are made to depend upon the arbitrarily defined units in the simplest way, so that calculations carried out in accordance with the fundamental rules of geometry or physical science lead to numerical answers giving magnitudes in the units of the system. For example, the foot of length and the cubic-foot of volume form part of an absolute system of units, because when the volume of a rectangular tank 4ft. by 3 ft by 2 ft. is calculated by the rules of mensuration, the product $4 \times 3 \times 2 = 24$ is the answer in units of the system, or in cubic feet. The foot and the gallon are not units of an absolute system because the answer to the sum, 24, has no simple arithmetical relation to the volume of the tank in gallons

**Fundamental and Derived Units.** The arbitrarily defined units of an absolute system are known as the fundamental units. Those units which are made to depend on the fundamental units are known as derived units. We have already stated that, in an absolute system, the number of fundamental units is as small as possible, and, for the measurement of geometrical and dynamical quantities, at least three of such fundamental units have been found necessary. The choice of the three fundamental units to be arbitrarily defined is, nominally, unrestricted, but, practically, the choice is made in such a way that the units can be specified with exact precision, that they can be copied or reproduced anywhere, and that they can readily be subdivided. The fundamental units chosen in some absolute systems are those of the physical entities, length, mass and time, which are intuitively felt to be of a fundamental character. Of these three units, that of mass is capable of the most exact definition, because the mass of a body, or the quantity of matter in it, is independent of external conditions which may be variable within the limits of practical experience. The actual material of the standard of mass must be one that is not subject to chemical change, such as the metal platinum. A

definite lump of platinum is therefore an exactly defined unit of mass. Such a unit can readily be duplicated or reproduced and subdivided, by means of a lever balance.

An arbitrary absolute unit of length can be defined as the distance between two marks on a bar of some material, like that used for the unit of mass, that is not liable to chemical change. Such a length unit can be duplicated or copied and subdivided with accuracy. We note however that the length unit is not so precise as the mass unit. The mass of a lump of matter is independent of ordinary temperature changes, but the physical size of the lump, and hence the distance between two marks on a material bar, will vary if its temperature alters. Hence the unit of length has to be specified, not only by reference to a material bar, but also by the temperature at which the unit is correct, and completely to define a length unit, it is necessary also to define a stipulated temperature.

Mass, and length or extension in space, are objective properties of matter, so that the units of these quantities can be defined by material objects. Time or duration is a non-material entity, and a unit of time has to be defined in a manner quite different from that of the definitions of mass and length units. The basis of the practical definition of a time unit is a corollary of Newton's first law of motion, that the angular speed of a body rotating without restraint will be constant. The rotation of the earth on its axis is known to be free from all but the most minute restraint, so that the angular speed of rotation is, to an exceedingly high degree of approximation, constant. The natural time unit is the sidereal day, or the interval between two successive instants when a fixed star has a specified direction relative to the earth's surface. This time interval can be observed with great precision, and it can be subdivided by an instrument called a clock, the action of which depends either on the counting of the isochronous vibrations of a pendulum, or on the

measurement of the turning of the rotor of an electric motor maintained at a constant and known rotational speed by the control exercised by an isochronously-

units of length and mass may be made to depend upon the time unit. Thus the length unit can be defined as that of a simple pendulum which, at a definite spot on the earth's surface, has a periodic time which is connected in a specified way with the absolute time unit, because this periodic time depends *only on the length* of the pendulum and the value of $g$, the acceleration of a freely-falling body due to its weight. The objection to such a definition of the length unit is that the simple

tion time as that of a material pendulum can be found, the connection between the physical dimensions of a material pendulum and its vibration time is of a recondite character. It is therefore more convenient, on practical grounds, to define the length unit directly and arbitrarily rather than indirectly in terms of a time unit. Similarly it would be possible to define a mass unit in terms of a length unit, as that of a stipulated volume of a specified substance. The practical realisation of a mass unit derived in this way from a length unit would depend on the measurement of a volume and also upon the purity of the material occupying this volume. Further, such a mass unit would be definite and precise only if the temperature were specified. For these reasons, it is preferable, on practical grounds, to define a mass unit directly and arbitrarily, rather than indirectly, in terms of a length unit and a specified substance. Length, mass and time units in an absolute system are therefore fundamental units, each being independently and arbitrarily defined.

The derived units of an absolute system which depend upon the fundamental units fall into three classes; geometrical units based upon the fundamental unit of length; kinematical units based upon the fundamental units of length and time; and dynamical units based upon all three fundamental units. The geometrical derived units are those of area and volume, and are defined respectively as the area of a square having each side of unit length, and as the volume of a cube each edge of which is of unit length. The kinematical unit of speed is a displacement of unit distance in unit time, that of acceleration is that of unit change in speed in unit time. The derivation of the dynamical derived units is based upon Newton's second law of motion that impressed force on a moving body is proportional to its mass and to the rate of change of its speed or to its mass multiplied by its acceleration, and the derived dynamical units are so defined that this law takes the more precise form that the measure of impressed force is numerically equal to the measure of mass multiplied by the measure of acceleration. Thus, with a fundamental unit of mass, unit force is that which will accelerate unit mass at the rate of unit speed in unit time. From this definition the unit of work or energy immediately follows; unit force acting through unit distance. Unit power is unit work done in unit time.

We observe that the definition of unit force in the preceding paragraph is not the only one possible. An alternative, and at first sight more objective definition, would be the gravitational attraction of the earth on unit mass at a specified position on the earth's surface. It would be necessary to specify a position in the definition, because the gravitational attraction of a definite piece of matter varies with the latitude and with the height above datum sea-level. The gravitational attraction of the earth acting on a body free to fall gives rise to constant acceleration, the magnitude

of which is usually denoted by the symbol $g$, and in all absolute systems of units in actual use, $g$ has a numerical value different from unity. If the unit of force were defined in terms of gravitational attraction, the statement of Newton's second law, force = mass × acceleration, would not be true, for unit force acting on unit mass would produce, not unit acceleration, but an acceleration of magnitude $g_0$, where $g_0$ is the acceleration due to gravity at the place specified for the definition of the force unit. Newton's law would therefore have to be stated in the form, force = mass × acceleration $\div g_0$. It would be more complicated, and would involve a quantity $g_0$ that would have to be experimentally determined As the object of the definition of the derived units of an absolute system is that fundamental dynamical laws are stated in the simplest manner possible, unit force is defined as that which produces unit acceleration of unit mass, so that Newton's basic law can be stated in the simplest and most general form, force = mass × acceleration.

The weight of a body, or the gravitational attraction of the earth exerted on it, is a force which, in an absolute system of units, is numerically equal to $mg$, where $m$ is the mass in absolute units and $g$ is the acceleration due to gravity at the place where the body is weighed, for this force accelerates the mass by the amount $g$. The weight of a body is therefore proportional to $g$. The weight of a body has a precise numerical connection with its mass only if a definite position on the earth's surface is defined, at which the value of $g$ is known. We shall see the importance of this idea a little later in our consideration of an absolute system in which the fundamental units are differently chosen from the way described above.

**C.G.S. Absolute System of Units.** This is the system used in scientific calculations The unit of length is the centimetre, defined as $\frac{1}{100}$ of the legal metre which is the distance between the ends of a platinum rod at a

temperature of 0 degrees C. The unit of mass is the gram, defined as $\frac{1}{1000}$ of the legal kilogram which is the mass of a piece of platinum. The unit of time is the second, which is the 86,400th part of a mean solar day. A solar day is the time interval between two successive instants at which the centre of the sun's disc appears due south at one spot on the earth's surface. Owing to the facts that the speed of the earth in its orbit is not constant, and that the earth's axis is not perpendicular to the plane of the orbit, the actual length of the solar day varies considerably throughout the year. The mean or average value of the solar day during a year is connected with the invariable sidereal day by the relation that one sidereal day is equal to 23 hours, 56 minutes, 4·09 seconds of mean solar time.

The metre of length upon which the C.G.S. absolute system is based was intended to be the one ten-millionth part of the distance between the north pole and the equator on the earth's surface. The kilogram was intended to be the mass of 1000 cubic centimetres of water at 4 degrees C. These relations are now known to be only approximate, and the actual definitions of the metre and the kilogram, and hence of the centimetre and the gram, are arbitrary, and in terms of objective material objects.

The table on p. 8 gives the fundamental and derived units of the C.G.S. system.

The drawback of the C.G.S. system is the exceedingly small magnitudes of the dynamical units. The dyne of force is about equal to the one 28,000th part of the weight of an ounce. The erg of work is equivalent to that required to raise one thousandth part of an ounce through a distance of one seventieth of an inch.

M.K.S. System of Units. The fundamental units in this system are the metre of length, the kilogram of mass, and the second of time. The units of length and mass are identical with the legal units referred to in the definition of the C.G.S. units. The table on p. 8 gives

### C.G.S. System of Units

| *Unit* | *How defined* | *Distinctive Name* |
|---|---|---|
| | FUNDAMENTAL UNITS | |
| Length | $\frac{1}{100}$ of legal metre | Centimetre |
| Mass | $\frac{1}{1000}$ of legal kilogram | Gram |
| Time | $\frac{1}{86400}$ of mean solar day | Second |
| | DERIVED UNITS | |
| Area | Square centimetre | |
| Volume | Cubic centimetre | |
| Speed | Centimetre per second | |
| Accelera-tion | Centimetre per second per second | |
| Force | Accelerates 1 gram by 1 centimetre per second per second | Dyne |
| Work or Energy | One dyne acting through 1 centimetre | Erg |
| Power | One erg per second | |
| Stress | One dyne per square centimetre | |

### M.K.S. System of Units

| *Unit* | *How defined* | *Distinctive Name* |
|---|---|---|
| Length | The legal metre | Metre |
| Mass | The legal kilogram | Kilogram |
| Time | The C.G.S. second | Second |
| Area | Square metre | |
| Volume | Cubic metre | |
| Speed | Metre per second | |
| Accelera-tion | Metre per second per second | |
| Force | Accelerates 1 kilogram by 1 metre per second per second | Newton |
| Work or Energy | One newton acting through one metre | Joule |
| Power | One joule per second | Watt |

the fundamental and derived units of the M.K.S. system.

The 'newton' force unit of the M.K.S. system is roughly equal to the weight of a quarter of a pound, and is a much more convenient magnitude than the C.G.S. force unit. The energy and power units of the M.K.S. system are identical with those used by electrical engineers.

Auxiliary units of the M.K.S. system are those having a simple numerical relation to the defined units. Thus the kilowatt of power is equal to 1000 watts. The kilowatt-hour of energy is equal to $3600 \times 1000 =$ 3,600,000 joules.

**F.P.S. System of Units.** In this system the unit of length, the foot, is defined as one third of the British legal yard which is the distance between transverse lines on gold plugs in a bronze bar, at 62 degrees F. The unit of mass is the legal pound, which is that of a definite piece of platinum. The unit of time is the second, defined as in the C.G.S. system. The derived units are defined as in the C.G.S. and M.K.S. systems and these units are given in the following table:

F.P.S. System of Units

| *Unit* | *How defined* | *Distinctive Name* |
|---|---|---|
| Length | One third of a legal yard | Foot |
| Mass | The legal pound | Pound |
| Time | The C.G.S. second | Second |
| Area | Square foot | |
| Volume | Cubic foot | |
| Speed | Foot per second | |
| Acceleration | Foot per second per second | |
| Force | Accelerates one pound by one foot per second per second | Poundal |
| Work or Energy | One poundal acting through one foot | Foot-poundal |
| Power | One foot-poundal per second | |

The unit of force in the F.P.S. system is about equal to the weight of half an ounce. This system is never used in practical scientific or engineering work, and it appears only in books on theoretical mechanics.

**British Engineers' Gravitational System of Units.** This system of units differs radically from the three preceding systems in that the fundamental units are those of length, force and time, and that mass is a derived unit. The units of length and time are the same as in the F.P.S. system; the foot and the second, respectively. The derived units of area, volume, speed and acceleration are, likewise, identical with the corresponding units of the F.P.S. system. The fundamental unit of force in the Gravitational system of units is the gravitational attraction on the weight of a legal pound of matter at London. To distinguish between pounds weight of force in the Gravitational system and pounds of mass in the F.P.S. system, the force unit is generally denoted by the abbreviation lb. As in the other absolute systems, the derived force unit is so defined that Newton's second law can be stated in the form, force = mass × acceleration, so, in the Gravitational system, unit mass is defined as that which acted upon by unit force will be accelerated by unit amount, or by one foot per second per second, so that in the Gravitational units, force = mass × acceleration The unit of work or energy in the Gravitational system is 1 lb. of force acting through 1 foot, and is called the ft.-lb. The power unit is one ft.-lb. per second.

We must consider very carefully what the mass unit in the Gravitational system really is. Unit force is defined as the gravitational weight of a legal pound of matter at London. If such a legal pound falls freely, then the force of its own weight, of unit magnitude, will accelerate it by $g_0$ units, where $g_0$ is the acceleration of gravity at London, in Gravitational length and time units, or in feet per second per second. Let $m_1$ stand for the measure of the mass of a legal pound in derived

Gravitational units. Then as unit force accelerates $m_1$ units of mass by $g_0$ acceleration units, we have, from the law, force = mass × acceleration, $1 = m_1 \times g_0$, so that $m_1$, the mass of the legal pound, is $\frac{1}{g_0}$ Gravitational units. Thus the unit of mass in the Gravitational unit is equal to $g_0$ legal pounds, or very nearly 32·2 pounds.

We can put this matter another way. Acceleration, with constant force, is inversely as mass. Unit force accelerates a legal pound by $g_0$ feet per second per second. Therefore unit force will accelerate $g_0$ legal pounds by 1 foot per second per second, and the Gravitational mass unit is $g_0$ *times* the legal pound of mass. We note that the number $g_0$ is the acceleration of gravity at a definite place—London.

Suppose that we have a lump of matter of unknown mass in legal units at some place other than London, where the acceleration due to gravity, $g$, is different from $g_0$, its value at London. How is the mass of this lump of matter in Gravitational units to be found? In practice the legal mass of a body is found in one of two ways. The first is by a lever balance, and this method is a true comparison of masses. If the lump of matter is so found to have a mass of $n$ legal pounds, this will be its mass in legal pound units at London, for mass is independent of position on the earth's surface. The mass of the lump of matter in Gravitational units will therefore be $\frac{n}{g_0}$, where $g_0$ is the acceleration due to gravity at London. If however a spring balance is used to determine the mass, the instrument will give a reading corresponding, not to true mass, but to gravitational weight, or to lbs. of force. This weight is directly proportional to $g$, the acceleration due to gravity at the place where the weighing is done. A

so that, denoting the spring balance weight by $w$, we have $w = n \times \frac{g}{g_0}$ and $n = w \times \frac{g_0}{g}$. The mass in Gravitational units $\frac{n}{g_0}$ is therefore equal to $\frac{w}{g}$. Thus mass, in

The Gravitational unit of mass, $g_0$ pounds, is sometimes called the slug.

The following table gives the fundamental and derived units of this Gravitational system of units :

British Engineers' Gravitational System of Units

| *Unit* | *How defined* | *Distinctive Name or Symbol* |
|---|---|---|
| | FUNDAMENTAL UNITS | |
| Length | One third of a legal yard | Foot |
| Force | The gravitational attraction or weight of a legal pound of matter at London | lb. |
| Time | The C G S. second | Second |
| | DERIVED UNITS | |
| Area | Square foot | |
| Volume | Cubic foot | |
| Speed | Foot per second | |
| Acceleration | Foot per second per second | |
| Mass | Accelerated one foot per second per second by unit force | Slug |
| Work or Energy | One lb acting through 1 ft | ft.-lb |
| Power | One ft.-lb per second | |

We may emphasise, in concluding this section, two points about the British Engineers' Gravitational System of Units. First, it is a true absolute system: the three fundamental units of length, force and time are exactly and arbitrarily defined, as are the units of length, mass and time in the other absolute system, while the derived Gravitational units, including that of mass, are obtained by the simplest application of geometrical and dynamical principles. Secondly, the absolute character of the system depends upon the specification, in the definition of the force unit, not only of a piece of matter but also of a definite spot on the earth's surface, London. Strictly speaking, this spot should be defined precisely by latitude, longitude and height above a datum, but for all practical purposes London is sufficiently exact. The British Engineers' Gravitational System has often been criticised on the ground that it is lacking in the precision of a true system of units and that the mass unit is not a constant quantity: such criticisms seem to be due to a failure to apprehend the vital point that locality must be, and actually is, specified in the definition of the fundamental force unit. The Gravitational system of units can be justly criticised on the ground that the fundamental force unit can be reproduced only at one place, London. unless a local characteristic, the value of $g$, is known. But this drawback is of little practical consequence.

An important auxiliary unit of the Gravitational system is the horse-power, which is equal to 550 power units, or 550 ft.-lbs. per second.

Numerical Relations of Units. The fundamental time unit is the same in all the absolute systems that have been considered, the second. There are three different units of length, the centimetre, the metre, and the foot. The centimetre and the metre are simply related. 1 metre $=100$ centimetres, while the foot and centimetre are connected by the relation 1 foot $=$ 30·4797 cms. The two mass units, the gram and the

pound are connected by the relation 1 pound = 453·593 grams. The M.K.S. kilogram unit is 1000 grams. As the Gravitational mass unit is $g_0$ pounds, or 32·191 pounds, any unit of length or mass can be expressed numerically in terms of any other. A number which indicates the ratio of two units of the same kind is called a conversion factor. We shall see later that from

Angle. Angle is a quantity which enters into many

is a measure of an amount of turning or rotation. The geometrical unit of angle is the right angle, which is subdivided into 90 degrees, the degree being further subdivided either sexagesimally into 60 minutes each equal to 60 seconds, or decimally. A second unit of geometrical angle is the radian, defined as that between two radii of a circle which define an arc of the cir-

radians, one revolution being equal to $2\pi$ radians.

Angular speed is measured either in revolutions per second or in radians per second. Angular acceleration is likewise measured either in revolutions per second per second, or in radians per second per second.

# CHAPTER II

## ELEMENTARY THEORY OF DIMENSIONS

**Basic Ideas.** We have seen in the preceding chapter how, after fixing arbitrarily a suitable number of fundamental units, all the other units of an absolute system requisite for the measurement of dynamical quantities can be derived from these fundamental units. Every derived unit depends on one or more of the fundamental units, and the connection of the derived units and the fundamental units can be stated verbally by an enunciation of the basic principle by which the derived unit has been fixed. The connection of a derived unit with the fundamental units, or the manner in which the measurement of a physical quantity depends upon the fundamental units of measurement, can be expressed in an algebraic or symbolic formula, and as we shall see hereafter, such a symbolic notation is a powerful instrument for several kinds of practical calculations. A symbolic statement of this kind is called the dimensions of the physical quantity to which it refers, or of the unit by which this physical quantity is measured.

We can illustrate the basis of this idea by a simple concrete example. Suppose that the size of a rectangle is 4 length units by 2 of the same length units. Then, as we have seen, the measure of its area in the derived units of an absolute system will be $4 \times 2$ area units. This can be expressed in the following way:

$$4 \text{ length units} \times 2 \text{ length units} = 8 \text{ area units.}$$

Let the symbol $[L]$ stand for the phrase 'length unit' and $[A]$ for the phrase 'area unit', then the preceding equation can be put in the form:

$$4[L] \times 2[L] = 8[A]$$

and treating $[L]$ and $[A]$ as algebraic quantities

$$8[L] \times [L] = 8[L^2] = 8[A]$$

So that $[A] = [L^2]$, and this equation is a symbolic statement of the fact that the unit of area in an absolute system is that of a square, each side of which is one [illegible] $[M]$ and $[T]$, for mass and time are quite irrelevant to area measurement We can indicate the fact that area is independent of $[M]$ and $[T]$ symbolically by writing

$$[A] = [L^2] \times [M^0] \times [T^0] = [M^0 L^2 T^0]$$

for, by the rules of algebra, the zero-th power of any quantity is unity so that $[M^0]$ and $[T^0]$ are each equal to 1. The symbolic statement $[A] = [M^0 L^2 T^0]$ is called a dimensional equation : it shows how the unit of area depends upon the fundamental units of length, mass and time. The dimensions of area are therefore said to be $M^0$, $L^2$, and $T^0$.

A similar kind of argument can be used to show the dimensions of a derived unit of volume. If a rectangular tank measures 4 length units by 3 length units by 2 length units its volume will be

$$24 \text{ volume units} = 4 \text{ length units} \times 3 \text{ length units} \times 2 \text{ length units},$$

and, by letting the symbol $[V]$ stand for the phrase 'volume unit', this equation is the same as,

$$24[V] = 4[L] \times 3[L] \times 2[L]$$

so that, with the same convention that $[V]$ and $[L]$ can be treated as algebraic quantities,

$$[V] = [L^3].$$

As volume, like area, is quite independent of units of mass or time, a fuller statement of the dimensions of the volume unit is,

$$[V] = [M^0 L^3 T^0]$$

Dimensions of Physical Quantities. The underlying idea in the preceding simple arguments was the assigning of a symbol for the fundamental units of measurements in an equation or sum for the calculation of some concrete or objective quantity, and by the treatment of this symbol as an algebraic quantity. The resulting dimensional equation, from which actual sizes of the measurements have been eliminated, is not exactly an equation in the usual sense in which the sign of equality, $=$, stands for identity of magnitude. In the dimensional equation $[A]=[M^0L^2T^0]$, the sign of equality is a symbol indicative of the kind of dependence of a derived unit on the fundamental units of the system to which this derived unit belongs. By the application of this idea of symbolic representation of fundamental units, the dimensions of other derived units of an absolute system can be worked out. We have, so far, regarded the fundamental units of mass, length and time as dimensionally fundamental. In other words, the dimensional statements about derived units are made in terms of the symbols $[M]$, $[L]$ and $[T]$. This is customary, but not obligatory, and we shall return later to the matter of the choice of the units to which fundamental dimensional symbols are assigned.

The derivation of the dimensions of area and volume are so simple as to be almost trivial. The next simplest quantities measured by derived units are speed and acceleration. Let us consider the case of a body which moves through 4 length units in 2 time units. The computation of its average speed during this time interval will be represented by the equation

$$2 \text{ speed units} = \frac{4 \text{ length units}}{2 \text{ time units}}$$

and using the symbol $[v]$, for 'speed unit,' we have, as before,

$$2[v] = \frac{4[L]}{2[T]} = 2\frac{[L]}{[T]}\,.$$

Thus $[v]=\frac{L}{T}$, which is often expressed in the form $[v]=LT^{-1}$. As speed has nothing to do with mass, the complete statement of the dimensions of $[v]$ is,

$$[v]=[M^0LT^{-1}].$$

As the dimensional symbol $[T]$ is in the denominator of a fraction, or bears a negative index, we infer at once that if the time unit of measurement is, say, increased, the speed unit will be reduced proportionally, and vice versa.

The two equivalent statements of the dimensions of speed, $\frac{L}{T}$, and $LT^{-1}$ have different appearances, and the first or fractional statement is the easier to read by those unaccustomed to mathematical symbolism. The notation in which symbols in the denominators of fractions have negative indices is however superior for purposes of more advanced calculations, and the reader is advised to accustom himself to this index notation

If the speed of a body changes from say 4 units to zero in 2 time units, its average acceleration during this time period will be computed by the equation

$$2 \text{ acceleration units}=\frac{4 \text{ speed units}}{2 \text{ time units}}.$$

If $[a]$ stands for 'acceleration unit,' this will be equivalent to :

$$2[a]=\frac{4[v]}{2[T]},$$

by substituting the value of $[v]$ we have already obtained, we find

$$[a]=\frac{L}{T}\div T=\frac{L}{T^2}=LT^{-2}.$$

The dimensions of acceleration are therefore $LT^{-2}$, or

more fully, $M^0LT^{-2}$, since acceleration is independent of mass.

The dimensional equation or formula for the dimensions of acceleration is, perhaps, not quite so obvious as those for area, volume and speed. The fact that the square of $[T]$ appears in the denominator of the dimensional formula shows that in a statement of a measure of acceleration in terms of fundamental units, the time unit must be twice mentioned. Thus an acceleration will be stated as so many, say, feet per second per second. This latter part of the statement is sometimes abbreviated to ft./sec.$^2$, but of course a second squared, or a square second has no objective meaning as a square foot has.

If a body of, say, 4 mass units is accelerated by 2 units of an absolute system then, by Newton's second law, the force required to produce this acceleration is computed by the following equation

8 force units = 4 mass units × 2 acceleration units.

And, if $[F]$ stands for 'force unit,' by using the dimensional formula for acceleration unit already found, we have

$$[F]=[M]\times[a]=\frac{ML}{T^2}=MLT^{-2},$$

and the dimensions of force are said to be $\frac{ML}{T^2}$ or $MLT^{-2}$.

If 8 units of force act through 2 length units, the work done is found from

16 work units = 8 force units × 2 length units,

so that if $[W]$ stands for 'work unit'

$$[W]=[F]\times[L]=\frac{ML^2}{T^2}=ML^2T^{-2},$$

and the dimensions of work are $\frac{ML^2}{T^2}$ or $ML^2T^{-2}$.

As stress is force per unit area, we now see, without intermediate steps that the dimensions of stress are

$$\frac{[F]}{[A]}=\frac{ML}{T^2}\div L^2=\frac{M}{LT^2}=ML^{-1}T^{-2}.$$

Similarly, as power is work done in unit time the dimensions of power are

$$\frac{[W]}{[T]}=\frac{ML^2}{T^2}\div T=\frac{ML^2}{T^3}=ML^2T^{-3}.$$

The reader will have noticed that the dimensional formulae for the dynamical quantities, force, work, power and stress, are by no means as obvious as those for the geometrical and kinematic quantities, such as area and speed. The dimensional formula for work will repay a little thought at this stage. It is $\frac{ML^2}{T^2}$ and as the dimensions of speed $[v]$ are $\frac{L}{T}$, an equivalent statement of the dimensions of work is $[M]\times[v^2]$. This shows that a physical quantity, the mass of a body multiplied by the square of its speed, has the same

its speed and its mass, and if the speed and mass are denoted respectively by $v$ and $m$, then the quantity $mv^2$ has the same dimensions as work. If we could assume that physical quantities of the same inherent nature have the same dimensions, we could infer that the kinetic energy of a body is proportional to $mv^2$, for $mv^2$ is a physical quantity of the same inherent character as

its mass and speed, even though $m$ and $v$ are each measured in the units of an absolute system, for the work unit of such a system is defined in terms of an

accelerating force and a length unit; and not, as it might conceivably be, as equivalent to the kinetic energy of unit mass moving with unit speed. We shall return to the point later.

Dimensional Properties of Materials. Quantitative statements about the physical properties of material bodies, like geometrical, kinematical and dynamical quantities, depend generally on units of measurements, and this dependence can be expressed by dimensional formulae. Two of the simplest properties of a material are its density and its specific gravity at a stipulated temperature. The density of a substance is defined as the mass of unit volume. By an argument similar to that used in the previous section, we see that the dimensions of density are obtained as those of mass ÷ volume; or

$$[\text{density}] = \frac{[M]}{[V]} = M \div L^3 = ML^{-3}T^0.$$

The specific gravity of a substance is defined as the mass of a stipulated volume of the substance divided by the mass of the same volume of a reference substance, water for solid or liquids, air or hydrogen for gases. The dimensions of specific gravity are therefore $\frac{[M]}{[M]} = 1 = M^0$, or, more fully, $M^0L^0T^0$. In other words, specific gravity is independent of all units of measurement; it is said to be a quantity of zero dimensions, or, otherwise, to be a dimensionless quantity. The difference between density and specific gravity should be carefully noted. The density of say mercury, in the F.P.S. system of units, 849 pounds per cubic foot, is very different from its density in the C.G.S. system, 13·596 grams per cubic centimetre. Its specific gravity, 13·596, is however independent of units of measurement, and is the same whether the masses of equal volumes of mercury and water are measured in grams, pounds, slugs, or in terms of any other mass units.

The circumstance that the density of a body in C.G.S. units is, for all practical purposes, equal to its specific gravity is due to the non-essential fact that the C.G.S. unit of mass is very approximately equal to that of unit volume, one cubic centimetre, of water.

Elasticity is a property of materials which may be approximately defined as the ratio of stress to reversible strain. We have already seen that the dimensions of stress are $\frac{M}{LT^2} = ML^{-1}T^{-2}$. Strain is defined as the fractional change of volume or of shape. Volumetric strain therefore has the dimensions $\frac{[V]}{[V]} = 1 = M^0L^0T^0$, and it is a dimensionless quantity. Shear strain or fractional change of shape can likewise be shown to be dimensionless. As the dimensions of strain are represented by the non-dimensional number 1, the dimensions of elasticity or of $\frac{\text{stress}}{\text{strain}}$, will be the same as those of stress or $\frac{M}{LT^2} = ML^{-1}T^{-2}$.

Surface tension is a property of liquids defined as the force per unit length of the boundary of the surface of the liquid. The dimensions of surface tension are therefore those of a force divided by a length or $\frac{ML}{T^2} \div L = \frac{M}{T^2} = MT^{-2}$. We see at once that this dimensional formula is the same as that of a quantity defined as energy per unit area, for this latter quantity will have the dimensions $\frac{ML^2}{T^2} \div L^2 = \frac{M}{T^2}$. Therefore, surface tension can be defined as the energy per unit area of the surface of a liquid. The dimensions of surface tension are fully stated as $ML^0T^{-2}$, and the measure of surface tension is independent of the length unit.

The viscosity of a fluid is defined in terms of its flow

in parallel stream lines over a fixed horizontal surface, as the force per unit area required to maintain the flow, divided by the change in the speed of the fluid per unit change of the distance from the fixed surface. The dimensions of viscosity are therefore found from the expression :

$$\frac{\text{force}}{\text{area}} \div \frac{\text{speed}}{\text{distance}}$$

and, substituting the known dimensions of the factors of this expression we obtain the dimensions

$$\left[\frac{ML}{T^2} \times \frac{1}{L^2}\right] \div \left[\frac{L}{T} \times \frac{1}{L}\right] = \frac{M}{LT^2} \times T = \frac{M}{LT} = ML^{-1}T^{-1}.$$

The kinematic viscosity of a fluid is defined as its viscosity divided by its density. The dimensions of kinematic viscosity are therefore found by dividing $ML^{-1}T^{-1}$ by those of density, $\frac{M}{L^3}$, and are

$$\frac{M}{LT} \div \frac{M}{L^3} = \frac{L^2}{T} = L^2T^{-1}.$$

Choice of Fundamental Dimensions. The preceding calculations of the dimensions of various physical quantities are all made in terms of the fundamental dimensions of mass $M$, length $L$ and time $T$. These are the fundamental units in the C.G.S., the M.K.S. and the F.P.S. systems of units. There is no essential need for the fundamental dimensions in dimensional formula to be those of the fundamental units of any particular system. As fundamental units can be chosen arbitrarily, so long as they are explicitly and exactly defined, so fundamental dimensions can be arbitrarily chosen. Thus it would be quite feasible to make the fundamental dimensions those of force $F$, length $L$ and time $T$, to correspond to the fundamental units of the British Engineers' Gravitational System of units. With these fundamental dimensions, the derived dimensions of

energy and power would be respectively $[FL]$ and $[FLT^{-1}]$ and would be simpler than those derived above. Mass would, however, have derived dimensions, force ÷ acceleration, or $F \times \frac{T^2}{L} = FL^{-1}T^2$, while the dimensions of density or mass ÷ volume would be $\frac{FT^2}{L^4} = FL^{-4}T^2$. The use of the dimensions of mass, length and time, as fundamental dimensions, to agree with the *fundamental units of the scientific C.G.S. system*, has the advantage that, in the long run, it leads to the simplest dimensional formulae. Apart from this consideration, the use of $M$, $L$ and $T$ is intuitively felt to be naturally most fitting, because mass, length, and *time are the fundamental entities of physical science.*

**Dimensionless Quantities.** We have already seen that certain objective quantities relating to material substances are independent of the units of measurement, and, *having zero dimensions in terms of mass, length and time*, can be termed dimensionless. These quantities are specific gravity and strain, each measured as the numerical ratio of two similar dimensional quantities. A large number of quantities enter into engineering and physical calculations, which have this inherent property, that they are dimensionless, and are quite independent of units of measurement. Dimensionless quantities are of three principal kinds; conversion *factors or the ratio of the measures of the same quantity* in two different systems of units, dimensionless variable quantities like specific gravity and strain; and fundamental numerical constants

centimetres. The number 30.4797 is most simply regarded as the result of *measuring a foot length in* centimetre units, but it will indirectly be obtained as

the ratio of the measures of the foot and centimetre lengths in terms of any other unit. Thus these two lengths might each be measured in inch units; the ratio of these measures, 12 inches to 0·3937 will be equal to the basic factor 30·4797. So, the basic conversion factor in the statement that 453·593 grams is equal to 1 pound, would be obtained by comparing the measures of the pound and the gram of mass in, say, ounce units. Conversion factors are thus dimensionless numbers, and independent of units of measurement. For every unit in an absolute system there is a conversion factor which expresses its measure in the units of another absolute system. Thus the conversion factor for the units of power in the M.K.S. and the Gravitational systems of units is expressed implicitly in the statement that 550 ft.-lbs. per second is equivalent to 746 watts. We shall see in the next chapter how conversion factors for derived units are calculated from the basic factors applicable to the fundamental units.

There is another kind of conversion factor which enters into commercial calculations when, for the sake of convenience, two or more units of the same quantity are in use. Thus, in this country, several units of length are in use, ranging in magnitude from the inch to the mile, and a typical conversion factor applicable to this system of length units is contained in the statement that 1 mile = 1760 yards. Conversion factors of this kind enter only indirectly into scientific and engineering calculations because, once the unique unit of measurement has been chosen, all quantities must be expressed in terms of this unit, however large or however small they may be. In a scientific calculation based upon the C.G.S. system, for instance, all lengths must be expressed in centimetres, all masses in grams, and all time intervals in seconds, otherwise the answer to the calculation will not be in terms of the appropriate derived unit of the C.G.S. system.

We have seen that, although the dimensions of work

or energy are those of a mass multiplied by the square of a velocity, the kinetic energy of a body of mass $m$, moving with speed $v$, is not necessarily equal to $mv^2$ in the energy units of the system of units in which $m$ and $v$ are measured. Actually, kinetic energy is equal to $\frac{1}{2}mv^2$. The $\frac{1}{2}$ in this formula is an example of a fundamental dimensionless numerical constant, and the manner in which this constant is derived will be worth a little study. The kinetic energy of the body or the work it can do by the loss of its speed, can be calculated in two ways. The first is by the use of the calculus. If the speed $v$ is reduced, the force exerted by the moving body is $m$ multiplied by its negative acceleration, or $m\dfrac{dv}{dt}$, and the work done as the body moves through a distance $ds$ will be

$$m\frac{dv}{dt}\times ds = m\,dv\times\frac{ds}{dt} = mv\,dv.$$

The total work done by the body by the total change of speed from $v$ to zero will therefore be:

$$m\int_0^v v\,dv = \tfrac{1}{2}mv^2,$$

[illegible]

by acceleratic [illegible]
a numerical [illegible]
$v\,dv$; it is [illegible]
differential expression.

We can, however, derive the expression $\frac{1}{2}mv^2$ in another way, without using the calculus. Assume that the body loses its speed at a constant time rate, and that the time for the whole loss of speed is $t_1$, during which the body moves through a distance $s_1$. The constant acceleration is $\dfrac{v}{t_1}$, and the constant force exerted through

the distance $s_1$ will be $\frac{mv}{t_1}$, so that the work done by the body in losing its speed will be $\frac{mvs_1}{t_1}$. But $\frac{s_1}{t_1}$ is the average speed during the time $t_1$, and as the speed decreases at a uniform rate the average will be equal to one half of the initial speed $v$. Thus $\frac{s_1}{t_1} = \frac{1}{2}v$, so that the work done is equal to $\frac{1}{2}mv^2$. The $\frac{1}{2}$ in this formula is the ratio of the average to the initial speed when the actual speed changes at a uniform rate. This $\frac{1}{2}$ is of the same character as the $\frac{1}{2}$ in the formula for the area of a triangle, $\frac{1}{2}$ the base multiplied by the height, or the $\frac{2}{3}$ in the corresponding formula for the area of a parabolic segment. Regarded as a constant of integration, the $\frac{1}{2}$ is a pure number, independent of all objective considerations, and depending only on the algebraic form of the differential expression $v\,dv$.

$\pi$ is a number which enters largely into scientific and engineering calculations. It is usually defined objectively as the ratio of the lengths of the circumference and the diameter of a circle, and according to this definition, $\pi$ is the ratio of two quantities of the same character. Actually, however, $\pi$ can be regarded as a pure number because, numerically, it is the sum to infinity of the following convergent series:

$$4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots\right)$$

where it is expressed in terms of the natural and abstract integers.

$e$ is another dimensionless quantity of frequent occurrence in scientific calculations. $e$ is usually defined as a pure number, and as the sum to infinity of the series:

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

but $e$ can also be defined geometrically in terms of the

rectangular hyperbola as $\pi$ is in terms of the circle. If the equation to the hyperbola is $xy=1$, then $e$ is the abscissa, the ordinate of which, with the ordinate $x=1$, the curve, and the horizontal axis, defines unit area.

Fundamental numerical constants are independent of units of measurements and are dimensionless. The $\frac{1}{2}$ in the kinetic energy formula is independent of whether the measure of energy is ergs in the C.G.S. system, or ft.-lbs. in the British Engineers' Gravitational system. We shall see later that not all numbers entering into engineering formulae are of this non-dimensional character and that one of the important applications of dimensional theory is to distinguish between the dimensionless constants and the dimensional numbers, which are dependent upon the system of units, and are therefore valid for one system only.

**Angular Magnitudes.** Angle, in its kinematic sense of amount of turning, is, as we have seen, commonly measured either in revolutions or radians. If a straight line turns about one fixed end, the angle described in radians is equal to the distance traversed by the moving end divided by the length of the line, the same unit of length being used for each measurement. The dimensions of angle therefore correspond to those of the fraction $\frac{\text{distance}}{\text{length}}$, and these dimensions are those of $\frac{L}{L}=1$, and are consequently all zero.

Angle is therefore a dimensionless quantity, independent of units of mass, length, and time. The units [illegible] radian, are in [illegible] ngle differs in [illegible] quantities; it is inherently a variable, in its kinematic sense, and its measure, although independent of mass, length and time units, depends upon the particular fundamental angular unit used for the measurement.

Angular velocity or speed is defined as angular movement in unit time, and its dimensions are those of $\frac{\text{angle}}{\text{time}}$, or, as the dimensions of angle are zero, $\frac{1}{T}=T^{-1}$, or, more fully, $M^0L^0T^{-1}$. The usual units of angular speed are the revolution per second and the radian per second.

The number of periodic occurrences that occur in unit time is known as the frequency. Since a number of occurrences is counted, this number is essentially dimensionless. $\frac{1}{T}$ or $T^{-1}$ therefore also gives the dimensions of a frequency.

Angular acceleration is the change of angular speed in unit time. Its dimensions are those of the fraction $\frac{\text{angular speed}}{\text{time}}$, and are therefore $\frac{1}{T}\div T$ or $\frac{1}{T^2}=T^{-2}$, or more fully $M^0L^0T^{-2}$. The usual units of angular acceleration are the revolution per second per second, or the radian per second per second.

The phase of a periodic quantity is the fraction of the periodic time that has elapsed since an arbitrarily fixed instant. The dimensions of phase are therefore those of $\frac{\text{time-interval}}{\text{periodic time}}$, or $\frac{T}{T}=1$. Phase is dimensionless like angle, and is often assessed quantitatively in angular measure on the basis that one period in time is equivalent in angle to one revolution, or $2\pi$ radians.

As the measure of angle is the ratio of a circumferential length to the radius of a circle, so the trigonometrical functions of angle, the sine and the cosine, are ratios of the length of straight lines defined by the angle and the radius. The sine and cosine functions are essentially dimensionless. The sine function of an angle $\theta$ is defined algebraically as the sum to infinity of the series:

$$\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}\cdots.$$

As $\theta$ is dimensionless, all its powers must be dimensionless also, so that the algebraic expression for $\sin \theta$ is dimensionless. The functions $\sin x$, and $\cos x$, occur in mathematics irrespective of any objective angular significance of the independent variable $x$. If, however, such an expression as $\sin x$ occurs in a mathematical equation representing some objective physical condition, $x$ must be a dimensionless quantity. Otherwise, the algebraic expression for $\sin x$ shows that it would be the sum of a number of terms, the dimensions of which would all be different.

**Dimensional Constants.** There are certain fundamental constant numbers which enter into scientific calculations that are essentially dimensional in character. A familiar example is the quantity $c$, the velocity of light, which enters into the formula for calculating the capacitance in microfarad units of an electrical condenser from its dimensions The dimensions of $c$ are, of course, those of a velocity or speed and are $\frac{L}{T}$ or $LT^{-1}$ The density of a body at a fixed temperature is another dimensional constant; its dimensions, as we have seen, are $ML^{-3}$

The gravitational constant is another important quantity of this class According to Newton's law of universal gravitation the attractive force between two bodies is proportional to the product of their masses and inversely proportional to the square of the distance between them. We therefore have this equation:

$$\text{force} = G \times \frac{\text{mass} \times \text{mass}}{(\text{length})^2}$$

where $G$ *is the factor which, when mass and length* are measured in the units of an absolute system will give the force in units of the same system. If we substitute the dimensions of force, mass and length, we obtain:

$$\frac{ML}{T^2}=G\times\frac{M^2}{L^2},$$

so that, $$G=\frac{L^3}{MT^2}=M^{-1}L^3T^{-2}.$$

Since the gravitational attraction of a gram of mass will accelerate it by 981 centimetres per second per second, this gravitational attraction is 981 dynes of force in the C.G.S. system. The gravitational attraction of the earth on the gram of mass is very approximately as if the mass of the earth were concentrated at its centre, so we can write the following equation:

$$980 \text{ dynes}=G\times\frac{1 \text{ gram}\times\text{mass of earth in grams}}{(\text{radius of earth in cm.})^2}.$$

From this equation we can, knowing the mass and radius of the earth, calculate $G$ in C.G.S. units, or knowing $G$ and the radius of the earth we can calculate the mass of the earth. $G$ has been determined experimentally and found to have the value $6{\cdot}664\times10^{-8}$ in C.G.S. units. By substituting this value of $G$ in the last equation, the mass of the earth is found to be approximately $6{\cdot}0\times10^{27}$ grams.

The following table contains no entries of quantities which occur only in rotational dynamics. We shall see later that the determination of the dimensions of these quantities gives rise to some difficulties which call for explanation and discussion.

## Table of Dimensions

| *Quantity* | *Defined as* | *Dimensions* Fractional Notation | *Dimensions* Index Notation |
|---|---|---|---|
| Mass | | $M$ | $M$ |
| Length | | $L$ | $L$ |
| Time | | $T$ | $T$ |
| Area | Length × length | $L^2$ | $L^2$ |
| Volume | Length × area | $L^3$ | $L^3$ |
| Speed | Space ÷ time | $\frac{L}{T}$ | $LT^{-1}$ |
| Acceleration | Speed ÷ time | $\frac{L}{T^2}$ | $LT^{-2}$ |
| Angle | Arc ÷ radius | 1 | 1 |
| Angular speed | Angle ÷ time | $\frac{1}{T}$ | $T^{-1}$ |
| Angular acceleration | Angular speed ÷ time | $\frac{1}{T^2}$ | $T^{-2}$ |
| Force | Mass × acceleration | $\frac{ML}{T^2}$ | $MLT^{-2}$ |
| Momentum | Mass × speed | $\frac{ML}{T}$ | $MLT^{-1}$ |
| Work or energy | Force × distance | $\frac{ML^2}{T^2}$ | $ML^2T^{-2}$ |
| Power | Work ÷ time | $\frac{ML^2}{T^3}$ | $ML^2T^{-3}$ |
| Stress | Force ÷ area | $\frac{M}{LT^2}$ | $ML^{-1}T^{-2}$ |
| Strain | Volume ÷ volume | 1 | 1 |
| Density | Mass ÷ volume | $\frac{M}{L^3}$ | $ML^{-3}$ |
| Specific gravity | Mass ÷ mass | 1 | 1 |
| Surface tension | Force ÷ length | $\frac{M}{T^2}$ | $MT^{-2}$ |
| Viscosity | Force per unit area ÷ speed per unit distance | $\frac{M}{LT}$ | $ML^{-1}T^{-1}$ |
| Kinematic viscosity | Viscosity ÷ density | $\frac{L^2}{T}$ | $L^2T^{-1}$ |
| Gravitational constant | Force × length² ÷ mass² | $\frac{L^3}{MT^2}$ | $M^{-1}L^3T^{-2}$ |

# CHAPTER III

## APPLICATIONS OF THE THEORY OF DIMENSIONS

**Basic Principles.** In this chapter we shall deal with some of the simpler practical applications of dimensional theory. These applications may be classified under three heads; the calculation of conversion factors, the checking of formulae, and the deduction of the nature of physical laws. Applications of dimensional theory depend upon two basic principles or axioms; first, that the dimensions of a physical quantity can be stated in one way and one way only in terms of the fundamental dimensional entities of mass, length and time; secondly, that an algebraic equation or formula relating to a physical quantity must be dimensionally homogeneous, or, in other words, the dimensions of every term of the equation must be identical. Thus, if a formula gives, say, a quantity of work as a function of some other quantities, then every term of the formula, or in the right-hand side of the equation, must have the dimensions of work. A quantity of work cannot be equated to the sum of a number of quantities of a different character. We shall see later that although the dimensions of a physical quantity can be stated in one way only, the converse of this principle is not always true, and that it is possible for quantities quite different in character to have identical dimensions in terms of mass, length and time, so that to indicate in a dimensional formula the essential difference in the character of such quantities some additional auxiliary dimension must be used.

**Conversion Factors.** We have already given some examples of conversion factors that are the numerical

ratios of fundamental unit quantities in different systems of units Dimensional theory points directly to the method of calculating the numerical ratios of derived units of the two systems. Suppose that a quantity has the dimensions $M^aL^bT^c$ and its measure in one system of units, $A$, is the number $K$ of the derived units. This as we have seen can be expressed symbolically as $K \times M^aL^bT^c$. Suppose now that the fundamental units of mass, length and time in a second system $B$ are denoted by $M_1$, $L_1$ and $T_1$, and that the mass unit in $A$ system is $n_1$ times that of the mass unit in $B$ system so that $M = n_1M_1$, and that, likewise, $L = n_2L_1$ and $T = n_3T_1$. Then we have :

$$K \times [M^aL^bT^c] = K \times [n_1M_1]^a \times [n_2L_1]^b \times [n_3T_1]^c$$
$$= K \times n_1^a n_2^b n_3^c \times [M_1^a L_1^b T_1^c]$$

The measure of the quantity in the $B$ system of units is, therefore, $n_1^a n_2^b n_3^c$ times the measure in the $A$ system. $n_1^a n_2^b n_3^c$ is the conversion factor giving the number of the derived units of the $B$ system that correspond to 1 of the same derived unit of the $A$ system.

Let us illustrate this general result by one or two simple numerical examples What is the equivalent of a speed of 60 miles per hour in ft. per second? Here the dimensions of speed are $\frac{L}{T}$. As 1 mile = 5280 ft, $n_2 = 5280$ Similarly, $n_3$, the number of seconds in 1 hour, is 3600. The required conversion factor is $\frac{5280}{3600} = \frac{88}{60}$, so that 60 miles per hour $= 60 \times \frac{88}{60} = 88$ ft. per second.

If the value of $g$ in British Engineers' units is 32·2 at London, what is its value in C.G.S. units? Here the dimensions of acceleration are $\frac{L}{T^2}$. As 1 ft. = 30 48 cm., $n_2 = 30\ 48$ The time unit in the two systems is the same, the second, so that $n_3 = 1$. The conversion factor for acceleration is therefore 30·48, so that 32·2 ft. per

second per second $= 32 \cdot 2 \times 30 \cdot 48 = 981$ cms. per second per second.

What is the factor for converting ft.-lbs. to ergs? Here the quantity considered is work, the dimensions of which are $\frac{ML^2}{T^2}$. The unit of mass, the slug, in the British Engineers' System is 32·2 pounds or $32 \cdot 2 \times 453 \cdot 6$ grams, and this is the value of $n_1$. $n_2$, the number of centimetres in 1 foot, is 30·48. The time unit is the same for both systems so that $n_3 = 1$. The required conversion factor is, therefore:

$$\begin{aligned}\frac{n_1 n_2^2}{n_3^2} &= 32 \cdot 2 \times 453 \cdot 6 \times (30 \cdot 48)^2 \\ &= 3 \cdot 22 \times 4 \cdot 536 \times (3 \cdot 048)^2 \times 10^5 \\ &= 135 \cdot 6 \times 10^5 \\ &= 1 \cdot 356 \times 10^7,\end{aligned}$$

so that

$$1 \text{ ft.-lb.} = 1 \cdot 356 \times 10^7 \text{ ergs.}$$

What is the equivalent of 1 h.p. in watts? Since 1 h.p. = 550 ft.-lbs. per second, we have:

$$\begin{aligned}1 \text{ h.p.} = 550 \text{ ft.-lbs. per sec.} &= 550 \times 1 \cdot 356 \times 10^7 \text{ ergs per sec.} \\ &= 550 \times 1 \cdot 356 \text{ joules per sec.} \\ &= 746 \text{ watts,}\end{aligned}$$

since 1 joule $= 10^7$ ergs, and 1 watt = 1 joule per second.

The arithmetic of conversion factors is dealt with in most elementary text-books on engineering mathematics and we need spend no further time on this subject. We may note in passing that the numbers in the British tables of weights and measures are, in a sense, conversion factors, although they apply to a single system of units. Thus 3 is the factor for converting yards to feet, and 112 is the factor for converting cwt. to pounds. When a physical quantity can be measured in two systems of commercial units as in cubic inches or cubic feet,

and in gallons and pints, we have two kinds of conversion factors, the first like the number 1728 for converting cubic feet into cubic inches applicable to one system, and the second like 6·229, for converting cubic feet in one system to gallons in the other.

The following table gives the principal conversion factors from the British Engineers' units to those in the C.G S. system .

Conversion Factors. British Engineers' to C.G.S. units

| *Quantity* | *Relevant dimensions* | *Factor* |
|---|---|---|
| Length, Velocity and Acceleration | $L$ | 30 48 |
| Area | $L^2$ | 929 03 |
| Volume | $L^3$ | 28316 |
| Mass | $M$ | 32·2 × 453 6 |
| Force and Momentum | $ML$ | 32 2 × 13824 |
| Work and Energy | $ML^2$ | 32·2 × 421·370 |

**Formulae.** We shall now give some illustrations of the second of the basic principles referred to above, that every term in an equation or formula referring to physical quantities must be of the same dimensions. If a body is projected vertically upward with an initial speed $v$, its height $h$, at a time $t$ measured from the instant of projection, is given by the formula :

$$h = vt - \tfrac{1}{2}gt^2,$$

where $g$ is the acceleration due to gravity. The dimensions of $v$ are $\frac{L}{T}$ and of $g$, $\frac{L}{T^2}$. Substituting the known dimensions, and representing the dimensions of the number $\frac{1}{2}$ by $[\frac{1}{2}]$, we have :

$$L = \frac{L}{T} \times T - [\tfrac{1}{2}] \times \frac{L}{T^2} \times T^2$$
$$= L - [\tfrac{1}{2}]L$$

The dimensions of $[\frac{1}{2}]$ are 1 or $M^0L^0T^0$. $\frac{1}{2}$ is therefore

a dimensionless number, and independent of the system of units in which $h$, $v$, $t$ and $g$ are measured.

It should now be noted that from the above short investigation, two kinds of deductions can be made. If we assume the truth of the formula for one absolute system of units, we have shown that the $\frac{1}{2}$ is a dimensionless number so that the formula is also true for any other absolute system, provided, of course, that $h$, $v$, $t$ and $g$ are all measured in the units of one system. If we assume that the $\frac{1}{2}$ is a dimensionless number we have checked the dimensional homogeneity of the formula and we can infer that it is probably correct.

Let us consider a more complicated example. The volume per second $V$ of liquid escaping from the end of a long tube is given by the formula

$$V=\frac{\pi}{8}\times\frac{pr^4}{l\eta},$$

where $p$ is the pressure difference along the tube of length $l$ and radius $r$, and $\eta$ is the viscosity of the liquid. The dimensions of $V$ are $\frac{L^3}{T}$, those of $p$ are force divided by area or $\frac{ML}{T^2}\times\frac{1}{L^2}$, and those of $\eta$, $\frac{M}{LT}$. Representing the dimensions of the numerical factor by $\left[\frac{\pi}{8}\right]$ we have, substituting the dimensional expressions

$$\begin{aligned}\frac{L^3}{T}&=\left[\frac{\pi}{8}\right]\times\frac{ML}{T^2}\times\frac{1}{L^2}\times\frac{L^4}{L}\times\frac{LT}{M}\\&=\left[\frac{\pi}{8}\right]\times\frac{L^3}{T}.\end{aligned}$$

The quantity $\frac{pr^4}{l\eta}$ therefore has the dimensions $\frac{L^3}{T}$, those of $V$. $\frac{\pi}{8}$ is therefore a dimensionless number independent of the system of units in which $p$, $r$, $l$ and $\eta$ are measured. When a numerical constant in a formula is a dimensionless number, the formula is correct for

all absolute systems of units provided that the quantities entering into the formula are all measured in the units of the system. Thus, if the formula just considered is used with British Engineers' units, $V$ would be in cubic feet per second, $r$ and $l$ would be in feet, $p$ in lbs. per sq. foot, and $\eta$ in slugs per foot-second.

Let us now consider a formula of a different class from the foregoing. Suppose that the height $h$ at time $t$ of a body projected vertically upward with an initial velocity $v$ is given as

$$h = vt - 16{\cdot}1t^2.$$

Substituting known dimensions, and denoting the dimensional expression of the numerical factor by [16·1], we have

$$L = \frac{L}{T} \times T - [16{\cdot}1]T^2$$

$$L = L - [16{\cdot}1]T^2.$$

We see at once that the factor 16·1 cannot be a dimensionless number, for if it were, we should have the absurd statement that a length is equal to the square of a time period. 16·1 is therefore a dimensional constant and the dimensions must be such that the term $16{\cdot}1t^2$ has the dimension $L$. The dimensions of [16·1] must therefore be $\frac{L}{T^2}$, those of an acceleration, and as a matter of fact we know that the factor 16·1 is equal to $\frac{1}{2}g$, where $g$ is measured in F.P.S. or in British Engineers' units. The formula, if true at all, as it actually is, is true only for particular fundamental units of length and time. The specification of the meanings of the symbols $h$, $v$ and $t$ given above is therefore insufficient, and with

*$v$ the initial velocity in feet per second, and $t$ the time in seconds*, and the formula is true only with this

reservation relating to the units employed. In contrast to this, the more general formula, quoted above, $h = vt - \frac{1}{2}gt^2$, is true for all absolute systems of units.

If we wish to obtain a formula equivalent to $h = vt - 16{\cdot}1t^2$, but applicable to the C.G.S. system in which $h$ is in cm. and $v$ in cm. per second, we must obtain the new value of the numerical factor 16·1 by means of the appropriate conversion factor. As 16·1 has the dimensions of an acceleration, $\frac{L}{T^2}$, this factor, as we have seen, is 30·48, so that the new formula is

$$\begin{aligned} h &= vt - 30{\cdot}48 \times 16{\cdot}1t^2 \\ &= vt - 491t^2, \end{aligned}$$

where $h$ is in cms., $v$ in cms. per sec., and $t$ in seconds.

Consider another formula given in the hand-books. The minimum thickness, $t$, of the header of a water-tube boiler at the tube holes for a tube diameter $d$, is

$$t = \tfrac{3}{32}\sqrt{d} + \tfrac{1}{4}$$

where $t$ and $d$ are in inches. $t$ and $d$ each have the dimension $L$, so that, indicating the dimensional expression for the numerical factors by $[\frac{3}{32}]$ and $[\frac{1}{4}]$, we have this dimensional equation

$$L = [\tfrac{3}{32}]\,L^{1/2} + [\tfrac{1}{4}].$$

As the dimensions of every term must be $L$, we see that the dimensions of $[\frac{3}{32}]$ are $L^{1/2}$, and of $[\frac{1}{4}]$ are $L$. The formula is true, therefore, only for the units specified. If we wish an equivalent formula in which $t$ and $d$ are in centimetres, then as 2·54 cm. = 1 inch, 2·54 is the conversion factor for $L$. The term $\frac{3}{32}$ in the new formula becomes $\frac{3}{32} \times \sqrt{2{\cdot}54}$, while the term $\frac{1}{4}$ becomes $\frac{1}{4} \times 2{\cdot}54$, so that, approximately,

$$t = 0{\cdot}15\sqrt{d} + 0{\cdot}62,$$

where $t$ and $d$ are each in centimetres.

We have already noted that if we can assume that

numerical constants in an equation are *dimensionless* numbers, we can obtain some sort of check of the truth of the equation by substituting the dimensional expressions for the variables and seeing that the dimensions of both sides of the equation agree. If we fail to obtain this agreement, we shall know that the equation or formula is untrue, or that it has been wrongly quoted. This principle is very useful for the discovery of mistakes when formulae are imperfectly remembered or copied wrongly from works of reference.

Let us consider a formula which purports to give the speed $v$ of a belt at which the maximum power is transmitted Suppose this formula is quoted as .

$$v=\sqrt{\frac{T_1}{3w}},$$

where $T_1$ is the maximum allowable tension, and $w$ the weight of unit length of the belt. Assuming the $3$ is a dimensionless number, and substituting dimensions $\frac{L}{T}$ for $v$, $\frac{ML}{T^2}$ for $T_1$ and, as weight is a force, $\frac{ML}{T^2}\times\frac{1}{L}$ for $w$, we have, squaring each side of the equation,

$$\frac{L^2}{T^2}=\frac{ML}{T^2}\times\frac{LT^2}{ML}=L.$$

*The dimensions do not agree, and to make the dimensions of the right-hand side of the equation the same as those of $v$, a factor having the dimensions $\frac{L}{T^2}$ must appear under the square-root sign.* A missing term therefore has the dimensions of an acceleration, *and we are reminded* at once that the speed for maximum power depends upon the mass per unit length of belt and not on the weight per unit length, and as mass is equal to weight $\div g$, the correct formula is

$$v=\sqrt{\frac{T_1 g}{3w}}.$$

The periodic time $t$ of a body of mass $m$ oscillating under the control of a force having a value $F$ per unit of displacement from the position of rest is known to be given by one of the formulae, $t=2\pi\sqrt{\frac{m}{F}}$, or $t=2\pi\sqrt{\frac{F}{m}}$. Which is correct? The dimensions of $F$ or force divided by length are $\frac{M}{T^2}$, and the quantity under the square-root sign must have the dimensions $T^2$. We see at once that, to satisfy this condition, this quantity must be $\frac{m}{F}$, so that the correct formula is

$$t=2\pi\sqrt{\frac{m}{F}}.$$

As a further example, consider the formula $T_1=\frac{wl}{8d}$ for the tension $T_1$ in the horizontal part of a wire suspended between two supports with a small sag $d$, where $l$ is the distance between the supports and $w$ the weight of unit length of the wire. The dimensions of $T_1$, a force, are $\frac{ML}{T^2}$ and those of $w$, a weight or force per unit length, are $\frac{M}{T^2}$. Substituting dimensions, and treating 8 as a dimensionless number, we have

$$\frac{ML}{T^2}=\frac{M}{T^2}\times\frac{L}{L}=\frac{M}{T^2}.$$

The dimensions of the two sides of the equation do not agree, and, therefore, the statement of the formula is wrong. We see at once that to make the dimensions of the expression on the right-hand side of the equation equal to those of a force, the formula must be $T=\frac{wl^2}{8d}$.

Finally, suppose the volume of water per second, $V$, flowing in a rectangular notch is known to be given by a formula $Kb\sqrt{2g}\times h^n$, where $K$ is a dimensionless

number, $b$ the breadth of the notch, $h$ the height of the water in the notch, $g$ the acceleration due to gravity, and $n$ an unknown index of $h$. The dimensions of $V$ are $\frac{L^3}{T}$, and substituting the dimensions of $b$, $\sqrt{g}$ and $h^n$, we have,

$$\frac{L^3}{T} = L \times \frac{L^{\frac{1}{2}}}{T} \times L^n.$$

If the dimensions of both sides of the equation are to agree, we must have

$$L^3 = L \times L^{\frac{1}{2}} \times L^n$$

or

$$3 = 1 + \tfrac{1}{2} + n,$$

so that $n = \frac{3}{2}$. The formula is therefore $V = Kb\sqrt{2g} \times h^{3/2}$.

**Deducing the Nature of Physical Laws.** One of the most important as well as the most interesting applications of dimensional theory is that of the deduction of the manner in which one physical quantity depends upon others. The usual method of discovering this kind of dependence or functional relationship is by deduction from simpler and more fundamental laws. In the dimensional method of deduction we write down a tentative equation which expresses, in the most general form, the dependence of one quantity upon several others, and we elucidate the nature of the dependence from the basic principle or axiom that the dimensions of each side of this equation must be in agreement. As dimensionless numbers cannot appear in dimensional equations, this method merely gives a statement of the proportionality of one quantity to powers of the others on which it depends: the dimensional method is incompetent to evaluate the numerical constants: it therefore reveals the nature of the law of dependence but it does not enable the law to be explicitly stated.

Suppose that a quantity $P$ is known or suspected to depend upon other quantities $Q$, $R$ and $S$. We can

express this dependence symbolically by an equation,

$$P = KQ^x R^y S^z,$$

where $K$ is a dimensionless number. If the dimensions of the quantity $P$ are $M^a L^b T^c$; those of $Q$, $M^{a_1} L^{b_1} T^{c_1}$; those of $R$, $M^{a_2} L^{b_2} T^{c_2}$, and those of $S$, $M^{a_3} L^{b_3} T^{c_3}$, then substituting these dimensions in the formula for $P$ we have

$$M^a L^b T^c = (M^{a_1} L^{b_1} T^{c_1})^x \times (M^{a_2} L^{b_2} T^{c_2})^y \times (M^{a_3} L^{b_3} T^{c_3})^z.$$

If the equation $P = KQ^x R^y S^z$ is in any sense true, the dimensions of each side of it must be identical, and if this condition is satisfied, the $M$ dimension on each side must be equal, so that

$$M^a = M^{a_1 x} \times M^{a_2 y} \times M^{a_3 z}$$

or

$$a = a_1 x + a_2 y + a_3 z.$$

Similarly, equating the dimensions of $L$ and of $T$ on the two sides of the equation, we have

$$b = b_1 x + b_2 y + b_3 z,$$
$$c = c_1 x + c_2 y + c_3 z.$$

In these last three equations, all quantities are known but $x$, $y$ and $z$, so that from the three simultaneous equations the unknowns can be determined by algebra. If, from this calculation, we find that either $x$, $y$ or $z$ are zero, this indicates that the quantity $P$ is independent of the quantity to which the zero index has been assigned. Thus, if in the original equation, or $P = KQ^x R^y S^z$, $z$ is found to be zero, then, as $S^0 = 1$, $P = KQ^x R^y$ and the value of $P$ is not affected by the value of $S$. We observe that in the above calculation the numerical constant $K$ disappeared in the dimensional equation, and that we are unable to make any inference about its magnitude.

Let us illustrate this rather abstract calculation by a number of actual examples. Consider first that the kinetic energy $E$ of a moving body may be assumed to

on the two sides of the equation, we find

$$0 = z,$$
$$1 = x + y,$$
$$-1 = -2y.$$

From which $y = \frac{1}{2}$, $x = \frac{1}{2}$ and $z = 0$, so that

$$v = K\sqrt{gh}.$$

The constant $K$ is found by other methods to be $\sqrt{2}$. As $z = 0$, the mass $m$ of the liquid is irrelevant, and $v$ is independent of this mass and depends only on $h$.

Consider the velocity $v$ of the propagation of a wave

$$v = Kf^x m^y a^z.$$

Substituting dimensions, we have, as those of $m$ are $ML^{-1}$.

$$LT^{-1} = [MLT^{-2}]^x \times [ML^{-1}]^y \times L^z.$$

Equating $M$, $L$ and $T$ dimensions, as before, we obtain:

$$0 = x + y$$
$$1 = x - y + z$$
$$-1 = -2x.$$

From the third and first equations, we find $x = \frac{1}{2}$ and $y = -\frac{1}{2}$ and, substituting these values in the second equation, we obtain.

$$1 = \tfrac{1}{2} + \tfrac{1}{2} + z,$$

so that $z = 0$. The required formula is therefore:

$$v = Kf^{1/2} m^{-1/2} = K\sqrt{\frac{f}{m}},$$

and $v$ is independent of $a$, the amplitude of the disturbance.

The energy of a vibrating and perfectly flexible

string is a constant quantity being the sum of the continually changing kinetic energy due to velocity, and potential energy due to displacement from the position of rest. This constant energy may depend upon the frequency of the vibration fixed by its tension and its mass per unit length, upon the amplitude of the middle point from the position of rest, and upon its total mass. Denoting frequency of dimensions $\frac{1}{T}$ by $f$, amplitude by $a$, and total mass by $m$, the dependence of total energy on these quantities can be thus stated:

$$E = Kf^x a^y m^z,$$

and substituting dimensions, we have:

$$ML^2T^{-2} = [T^{-1}]^x L^y M^z.$$

Equating dimensions $M$, $L$ and $T$, we obtain:

$$1 = z,$$
$$2 = y,$$
$$-2 = -x.$$

Or $E = Kf^2a^2m$. The energy of a vibrating string is therefore proportional both to the square of the frequency and to the square of the amplitude of the vibration.

As a further example, let us consider a raindrop falling through the air. The gravitational force accelerating the downward velocity of the drop will be opposed by a retarding force $F$, which may depend upon the velocity of fall $v$, the radius of the drop $r$, and upon the viscosity of the air $\eta$. Otherwise:

$$F = Kv^x r^y \eta^z.$$

Substituting dimensions, and recollecting that those of viscosity are $ML^{-1}T^{-1}$, we have:

$$MLT^{-2} = [LT^{-1}]^x \times [L]^y \times [ML^{-1}T^{-1}]^z.$$

Equating dimensions of $M$, $L$ and $T$, as before, we obtain:

$$1=z,$$
$$1=x+y-z,$$
$$-2=-x-z.$$

From the first and third of these equations, we obtain, $x=1$, $z=1$, and substituting these values in the second equation gives $y=1$. The required formula for $F$ in terms of $v$, $r$ and $\eta$ is, therefore.

$$F=Kvr\eta,$$

where $K$ is a dimensionless number. The precise equation is $F=6\pi vr\eta$ The gravitational force acting on the raindrop is proportional to its volume, or to $r^3$. The velocity of fall will be constant when the gravitational and retarding forces are equal, or when

$$K_1 r^3 = Kvr\eta,$$

where $K_1$ is independent of $v$, $r$, and $\eta$ Thus, the constant velocity of fall is proportional to $r^2$, and for very small values of $r$ this constant or limiting velocity becomes negligibly small, so that the drops float and form a cloud or mist.

The formula on page 37 for the rate of efflux of a liquid from the end of a long tube is established on the assumption of stream-line flow, that the velocity of the liquid is everywhere parallel to the axis of the tube This condition applies only when the velocity does not exceed a critical value. If the velocity is greater than this critical value the flow ceases to be stream-lined and becomes turbulent.

The critical velocity $v_1$ depends upon the viscosity of the liquid $\eta$, its density $\sigma$, and upon the radius of the tube $r$. Thus we may write,

$$v_1 = R\eta^x\sigma^y r^z,$$

where $R$ is a dimensionless number. Substituting the

dimensions of $v_1$, $\eta$, $\sigma$ and $r$, we obtain:

$$LT^{-1}=(ML^{-1}T^{-1})^x \times (ML^{-3})^y \times L^z,$$

which gives

$$0=x+y,$$
$$1=-x-3y+z,$$
$$-1=-x,$$

from which $\quad x=1, \quad y=-1, \quad z=-1,$

so that $v_1=R \times \dfrac{\eta}{\sigma r}$, where $R$ is a dimensionless number independent of the system of units employed. The formula shows that viscosity assists stream-line flow but density tends to prevent it. The dimensionless constant $R$ is sometimes called the Reynolds number.

Dimensions in Co-ordinate Geometry. Co-ordinate geometry is a branch of mathematics in which the properties of curves and surfaces are investigated by means of algebraic equations which express the functional relationships between the co-ordinates of every point on the curve or surface. These co-ordinates have an objective meaning. They are the distances of a point on the curve or surface from the axes or planes of reference, and, accordingly, the co-ordinates each have the dimensions of a length. As the equations of co-ordinate geometry likewise have an objective meaning, all the terms of such equations must have the same dimensions. If an equation in co-ordinate geometry is algebraically homogeneous, that is, if the sums of the indices of the factors of each term are all equal, then each quantity, whether variable or constant, will have the dimensions of a length. Otherwise, and if the equation is not homogeneous algebraically, some of the quantities will have a dimension other than that of a length.

Consider, for instance, the equation to a straight line in the co-ordinate geometry on a plane. This equation can be written in several ways. One form of the equa-

tion is $\frac{x}{a}+\frac{y}{b}=1$. This equation is algebraically homogeneous; the sum of the indices of the terms on the left-hand side is $+1-1$ or zero and this corresponds to the dimensions of the pure number 1 on the right-hand side. $a$ and $b$ therefore each have the dimensions of $x$ and $y$, that is, each stands for a length. Actually, of course, $a$ and $b$ are the intercepts in the horizontal and vertical axis determined by the curve.

Another form of the straight-line equation is $y=b+px$. Here, as $y$ has the dimension $L$, $b$ is a length. $px$ also has the dimension $L$, so that, as $x$ is a length, $p$ must be a dimensionless number, or the ratio of two lengths. If we convert the equation $\frac{x}{a}+\frac{y}{b}=1$, to $y=b-\frac{b}{a}x$, we see that $p$ is minus the ratio of the intercepts $b$ and $a$, it is the slope of the line.

$lx^2$ :

ho

hand terms is 3, while that of the right-hand terms is 1. $l$, $m$ and $n$ cannot therefore all stand for lengths. The more usual form of the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is algebraically homogeneous. $a$ and $b$ each have the dimensions of a length, and, as a matter of fact, they are the horizontal and vertical axes respectively of the closed curve.

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with the curve are coincident, then, if the statements of the basic equations are algebraically homogeneous, every intermediate equation in the calculation will also be homogeneous, and a partial check of the accuracy of the working can be obtained by inspection

at every step of the reasoning. If non-homogeneous equations are used, this check by inspection is not possible.

The expression or equation for the slope of a curve, obtained by differentiation or by algebra, must have every term of zero dimensions in length, for a slope is the ratio of the lengths of two lines at right angles. Similarly, the second differential coefficient $\frac{d^2y}{dx^2}$ of the equation to a curve has the dimensions of slope $\div$ length, or, as slope is dimensionless, of $\frac{1}{L}=L^{-1}$. The curvature of a circle is the reciprocal of its radius and has the dimensions $L^{-1}$. The second differential coefficient of $y$ in the equation to a curve therefore has the same dimensions as its curvature.

When the equation to a curve is given in a non-homogeneous algebraical form, the nature of the constant factors can often be inferred by recollecting the foregoing principles. Thus, suppose that the equation to a parabola is given as:

$$y=a+bx+cx^2.$$

As $y$ is a length, each term on the right-hand side must have the dimensions $L$. $a$ is therefore a length; it is the intercept cut off by the curve on the vertical axis of co-ordinates. As $bx$ has the dimensions $L$, $b$ must be a dimensionless factor; it is a slope, and, as when $x$ is exceedingly small so that $x^2$ can be neglected the equation becomes $y=a+bx$, we see that $b$ is the slope when $x=0$ and where the curve cuts the vertical axis. Similarly, $cx^2$ has the dimensions $L$, so that $L=cL^2$, and the dimensions of $c$ are $\frac{1}{L}$ or $L^{-1}$. $c$, therefore, has the dimensions of a curvature, and, as a matter of fact, $2c$ is the curvature at the point on the curve where the slope is zero.

# CHAPTER IV

## DIMENSIONAL THEORY OF THE DYNAMICS OF ROTATION

**Dynamics of Rotation.** In Chapter II we worked out the dimensions of the various quantities that occur in the simple branch of dynamics which may be termed the dynamics of a particle or the dynamics of translational motion, and we there ignored those quantities that occur in that more difficult branch of the whole subject of dynamics, the dynamics of rotating bodies Not only is this latter branch the more difficult, but, as we shall see and explain in the present chapter, the application of the theory of dimensions to it gives rise to difficulties and ambiguities which are not found in connection with the simpler branch of the subject considered in Chapter II. Before proceeding to the working out of the dimensions of the quantities occurring in the dynamics of rotation, it will be well for us to recall that the two branches of the whole subject of dynamics are strictly analogous, in that to every quantity in translational dynamics, there is an exactly corresponding quantity in rotational dynamics, and that to the basic laws of the translational dynamics there are the analogous laws of rotational dynamics. Thus, we can pass from the simpler to the more difficult branch of dynamics by a list of terms and statements of rules which forms a kind of dictionary. This dictionary is given in the following table :

| *Translational Dynamics* | *Rotational Dynamics* |
|---|---|
| Length or distance | Angle |
| Velocity | Angular velocity |
| Acceleration | Angular acceleration |
| Mass | Moment of inertia |
| Time | Time |

| *Translational Dynamics* | *Rotational Dynamics* |
| --- | --- |
| Momentum | Angular momentum or moment of momentum |
| Force | Couple or torque |
| Work and energy | Work and energy |
| *Force = mass × acceleration* | Torque = moment of inertia × angular acceleration |
| Work = force × distance | Work = torque × angle |
| Kinetic energy = $\frac{1}{2}$mass × (velocity)$^2$ | Kinetic energy = $\frac{1}{2}$ moment of inertia × (angular velocity)$^2$ |

In the above list we see that time, and work and energy are common to the two systems. We have already seen in Chapter II that the units of angular velocity and acceleration are respectively the radian per second, and the radian per second per second. Energy, work and time units are common to the two systems. We shall see that couple or torque is measured by force multiplied by distance, the 'arm' of the couple. The unit couple in the C.G.S. system of units is the gram centimetre; in the British Engineers' System, the unit couple is the lb.-ft., the term lb.-ft. being used to distinguish the unit couple from the unit of work which is the ft.-lb.

Dimensions of Quantities in Rotational Dynamics. The dimensions of the kinematic quantity angle, or amount of turning, and the derived quantities, angular velocity and angular acceleration have already been dealt with in Chapter II. We have there seen that as kinematic angle is measured in radians by the ratio of circumferential movement to radial distance, the dimensions of angle, in terms of $M$, $L$ and $T$ are $L \div L = 1$, or, symbolically, $M^0L^0T^0$. Angle is a quantity of zero dimensions, and it is therefore dimensionally similar to a pure number. Consequently, the quantity 'angle' cannot enter into an ordinary dimensional formula. The dimensions of angular

velocity and angular acceleration are as we have seen, $L^{-1}$ and $L^{-2}$ respectively.

The moment of inertia $I$, of a body in reference to an assigned axis is defined by the formula $\Sigma mr^2$ where $m$ is an infinitesimal element of mass and $r$ is the distance of this element from the assigned axis. Otherwise, $I$ is equal to the whole mass of the body multiplied by the square of the radius of gyration, which latter quantity depends upon the shape and size of the body and the position of the assigned axis relative to it. The dimensions of moment of inertia are therefore those of mass multiplied by (length)$^2$ or $ML^2$.

The angular momentum of a rotating body is the product of its moment of inertia about the axis of rotation by its angular velocity, or $I\omega$, where $\omega$ stands for angular velocity. The dimensions of angular momentum are therefore $ML^2 \times \frac{1}{T} = \frac{ML^2}{T} = ML^2T^{-1}$.

The moment of the momentum of a particle of mass $m$, moving in a circular path of radius $r$, with a circumferential velocity $v$, is defined as $mvr$. The dimensions of moment of momentum are therefore

$$M \times \frac{L}{T} \times L = ML^2T^{-1},$$

and these are the same as the dimensions of angular momentum.

Torque, couple, or the twisting effort of two coplanar and equal parallel forces of the same magnitude, but acting in opposite directions, is measured by the product of the measure of one of the forces by that of the distance between their directions The dimensions of torque or couple are therefore those of force × length

or $$\frac{ML}{T^2} \times L = \frac{ML^2}{T^2} = ML^2T^{-2}.$$

We can use the foregoing results to check the dimensional consistency of those laws or principles of the

dynamics of rotation, listed in the above table, which correspond to the basic principles of the simpler translational dynamics.

From torque = moment of inertia × angular acceleration we have, substituting dimensions,

$$\frac{ML^2}{T^2} = ML^2 \times \frac{1}{T^2} = \frac{ML^2}{T^2},$$

which is correct.

Similarly, torque × angle = work, we have, as angle is a dimensionless quantity in terms of $M$, $L$ and $T$, $\frac{ML^2}{T^2} = \frac{ML^2}{T^2}$. Lastly, from kinetic energy = $\frac{1}{2}$ moment of inertia × angular acceleration we have, substituting dimensions, $\frac{ML^2}{T^2} = ML^2 \times \frac{1}{T^2}$, which also is correct. The principal laws or rules of rotational dynamics are therefore dimensionally consistent.

Notwithstanding this consistency, there is one striking anomaly in the dimensions we have deduced for the quantities occurring in the dynamics of rotation: the dimensions of couple or torque are identical with those of work or energy, $ML^2T^{-2}$. In other words, two quantities essentially different in physical character have the same dimensions.

Now, although this anomalous result does not contradict the fundamental principle or axiom enunciated at the beginning of Chapter III, that the dimensions of a physical quantity can be stated in one way only, it does introduce an ambiguity into the dimensional formulae of the dynamics of rotation. In translational dynamics, a quantity having the dimensions $ML^2T^{-2}$ is recognised at once as standing for work or its equivalent, energy; in rotational dynamics, this dimensional formula is equivocal, it can stand either for energy or torque. The cause of this ambiguity attaching to the dimensional formula, $ML^2T^{-2}$, is very clear. In translational

represent variable or kinematic angle in terms of the fundamental dimensional entities, mass, length and time. The only way to bring this variable quantity into dimensional equations is to treat it as a fourth dimensional entity, and to assign to it a special dimensional symbol. Let us represent the new dimension, angle, by the symbol $[\psi]$. The dimensions of quantities in the dynamics of rotation will then be expressed in terms of four fundamental entities, $M$, $L$, $T$ and $\psi$. Using this new dimensional symbol, the dimensions of angular velocity and angular acceleration are respectively $[\psi T^{-1}]$ and $[\psi T^{-2}]$.

The assignment of a dimensional symbol to angle removes at once the ambiguity of the identical dimension of the dissimilar quantities torque and work. For as torque × angle is equal to work, we have the dimensional equation :

$$[\text{torque}] \times [\psi] = [ML^2T^{-2}],$$

whence the dimensions of torque are $\dfrac{ML^2}{T^2\psi}$ or $ML^2T^{-2}\psi^{-1}$.

We can now go forward and calculate the dimensions of other quantities in rotational dynamics in terms of the four fundamental entities, $M$, $L$, $T$ and $\psi$. As torque = moment of inertia × angular acceleration, we have the dimensional equation,

$$\frac{ML^2}{T^2\psi} = \text{moment of inertia} \times \frac{\psi}{T^2},$$

whence, the dimensions of moment of inertia, deduced from dynamical considerations, are $\dfrac{ML^2}{\psi^2} = ML^2\psi^{-2}$.

As angular momentum is equal to moment of inertia × angular velocity, we have, for the dimensions of angular momentum,

$$\frac{ML^2}{\psi^2} \times \frac{\psi}{T} = \frac{ML^2}{T\psi} = ML^2T^{-1}\psi^{-1}.$$

Let us check the consistency of the relation, kinetic energy of rotation $= \frac{1}{2}$ moment of inertia $\times$ (angular velocity)$^2$, in terms of the four dimensional entities. The dimensional equation is :

$$\frac{ML^2}{T^2} = \frac{ML^2}{\psi^2} \times \frac{\psi^2}{T^2} = \frac{ML^2}{T^2},$$

which is correct.

Let us now return to the problem considered in the previous section, and see whether the use of a dimensional symbol for variable angle sheds more light on the law determining the periodic time of torsional oscillations. We can now re-state the dimensions of the quantities in the tentative formula $t_0 = KI^xC^y\alpha^z$. The new dimensions of $I$ are $\frac{ML^2}{\psi^2}$, those of $C$, torque per unit angular displacement or torque divided by angle, are $\frac{ML^2}{T^2\psi} \times \frac{1}{\psi} = \frac{ML^2}{T^2\psi^2}$, and the dimensions of amplitude $\alpha$ are $\psi$. Thus, we have this dimensional equation :

$$T = [ML^2\psi^{-2}]^x \times [ML^2T^{-2}\psi^{-2}]^y \times [\psi]^z,$$

equating dimensions of $M$, $L$, $T$ and $\psi$, we obtain :

$$\begin{aligned} 0 &= x + y, \\ 0 &= 2x + 2y, \\ 1 &= -2y, \\ 0 &= -2x - 2y + z = -2(x + y) + z. \end{aligned}$$

From the third and first of these equations we obtain $y = -\frac{1}{2}$ and $x = +\frac{1}{2}$. From the fourth, by substituting $x + y = 0$, from the first, we obtain $z = 0$. Thus we can deduce, first that $t_0 = K\sqrt{\frac{I}{C}}$, as before, and secondly, what we were unable to deduce in the previous calculation, that $z = 0$, and that, consequently, the angular amplitude cannot enter into the formula for the periodic time $t_0$. Otherwise, $t_0$ is independent of the amplitude,

and consequently, the torsional oscillations are isochronous.

The Meaning of Angle in Dimensional Formulae. *Although the use of an additional dimensional symbol* for kinematic angle enlarges the scope of dimensional analysis as applied to the subject of rotational dynamics, we must not overlook the fact that, at first sight, this new symbol appears to have eliminated some inconsistencies at the expense of the introduction of others. Thus, although assigning the dimensions $ML^2T^{-2}\psi^{-1}$ to torque has rationalised the dynamical formula torque × angle = work, and has given a dimensional formula for torque which is different from that for work, the additional symbol seems to have introduced an inconsistency in the fundamental definition of torque as force × distance, for the dimensions of this product are $\frac{ML}{T^2} \times L$. If we consider that the dimensional formula $\frac{ML^2}{T^2\psi}$ represents in some way the product force × distance, *this can be shown as* $\frac{ML}{T^2} \times \frac{L}{\psi}$, *from* which it appears that, in the torque definition, force is multiplied by a quantity having the dimensions, not $L$, but $\frac{L}{\psi}$. We observe that, in terms of $M$, $L$, and $T$, the dimensions of $L$ and $\frac{L}{\psi}$ are identical, and that $\frac{L}{\psi}$ stands *for the dimensions of a length, as does the* $L$ *in the* dimensional expression $\frac{ML}{T^2}$ for force. What then is the objective significance of the dimensional expression $\frac{L}{\psi}$, *for the quantity by which force is multiplied to obtain* the measure of torque or couple? We can put this ques*tion in another form : what is the essential difference in* the two statements that force × distance is equal to

work, and that force × distance is equal to torque? The answer to this question is easily given. Distance, in the formula for work is in the same direction as the force; distance in the formula for torque is in a direction perpendicular to that of the force. This means that in the dimensional equation corresponding to the statement, torque = force × distance, or $\frac{ML}{T^2} \times \frac{L}{\psi} = \frac{ML^2}{T^2\psi}$, $\frac{L}{\psi}$ is a distance perpendicular to that of the $L$ in the dimensional formula $\frac{ML}{T^2}$ for force, which latter $L$, of course, corresponds in direction to the force or to the acceleration that it produces. In reference to the rotation produced by a torque, the $L$ in the force factor of the dimensional formula is a circumferential length, while the $\frac{L}{\psi}$ corresponding to the arm of a couple is in a perpendicular radial direction. We can justify the dimensional symbol, $\psi$, indicating perpendicular relation between two directions, for angle is measured as the ratio of a circumferential to a radial length. $[\psi]$ therefore has the dimensions, circumferential length ÷ radial length, so that, in reference to a circumferential length of dimensions $L$, a radial length should have the dimensions $\frac{L}{\psi}$, which is what we found from dynamical considerations. The dimensional formula $ML^2T^{-2}\psi^{-1}$ for torque is therefore a consistent one and it shows that, dynamically, torque can be measured in terms of work per radian of angular movement. The dynamical unit of torque in the British Engineers' System would then be 1 ft.-lb. per radian.

Again, consider the quantities angular momentum and moment of momentum. We have seen that the dimensions of the first quantity in terms of $M$, $L$, $T$ and $\psi$ are $\frac{ML^2}{T\psi}$. The moment of momentum of a rotating

particle has been defined as $mvr$, an expression into which angle does not enter and appears to have the dimensions $\frac{ML^2}{T}$. If, however, we reflect that $v$ is a circumferential velocity while $r$ is a radial distance, the argument in the preceding paragraph shows that one $L$ in the $\frac{ML^2}{T^2}$ formula is radial while the other is circumferential. As the circumferential $L$ is associated with the $M$ and the $T$, the radial distance $r$ should have the dimensions $\frac{L}{\psi}$, so that the correct dimensional formula for moment of momentum is $\frac{ML^2}{T\psi}$, which agrees with that for angular momentum.

Lastly, consider the centrifugal force exerted by a particle of mass $m$, constrained to move in a circular path of radius $r$, with a circumferential velocity $v$, and an angular velocity $\omega$. The centrifugal force can be calculated from three formulae, $f=\frac{mv^2}{r}$, $f=mv\omega$, and $f=m\omega^2r$. The dimensional equations corresponding to these formulae are, at first thought:

$$[f]=M\times\frac{L^2}{T^2}\times\frac{1}{L}=\frac{ML}{T^2},$$

$$[f]=M\times\frac{L}{T}\times\frac{\psi}{T}=\frac{ML\psi}{T^2},$$

$$[f]=M\frac{\psi^2}{T^2}L=\frac{ML\psi^2}{T^2},$$

and these dimensional equations are inconsistent. Here, however, as in the calculation of moment of momentum, we must take account of the fact that the radial distance $r$ is perpendicular to the direction of $v$, and accordingly $r$ must be given the dimensions $\frac{L}{\psi}$. If this substitution is made, the dimensions of $f$ work out

to be $\frac{ML\psi}{T^2}$ from all three equations, and dimensionally they become consistent. But a serious difficulty still remains. We have arrived at two distinct and different dimensional formulae for force, $MLT^{-2}$, and $MLT^{-2}\psi$, and this conflicts with the fundamental axiom or principle of dimensional analysis, that the dimensions of a physical quantity can be stated in one way only, unless we can assign some objective difference between a centrifugal force of dimensions $MLT^{-2}\psi$, and a force having the dimensions $MLT^{-2}$. A little reflection will, however, show that there is such a difference. In translational dynamics, a force accelerating a mass is conceived to change the magnitude of its velocity; a centrifugal force acting on a mass moving in a circular path continually alters the direction of the movement but leaves the magnitude of the velocity unchanged. The action of the first kind of force does work on the mass for it increases its velocity, and, hence, its kinetic energy. The centrifugal force, although acting continually on the mass, leaves its kinetic energy due to its circumferential velocity unchanged and, therefore, does no work. When the first kind of force acts through a definite distance, this is represented by the dimensional formula $\frac{ML}{T^2} \times L = \frac{ML^2}{T^2}$ which represents work, or energy. The second kind of force, or centrifugal force, in conjunction with circumferential movement is represented dimensionally by $\frac{ML\psi}{T^2} \times L = \frac{ML^2\psi}{T^2}$, and as the dimensional expression $\frac{ML^2}{T^2\psi}$ representing torque is essentially different from $\frac{ML^2}{T^2}$ representing energy or work, so is $\frac{ML^2\psi}{T^2}$. If a radial force acts on a body moving in a circular path, in opposition to centrifugal

force, so as to diminish the radius of the path, then work will be done, for as radial distance has the dimension $\frac{L}{\psi}$, this will correspond dimensionally to $\frac{ML^2\psi}{T} \times \frac{L}{\psi} = \frac{ML^2}{T^2}$, which is the dimensional expression for work. In this case, we know that the action of the force leaves the moment of momentum, $mvr$, unchanged, so that, as $r$ decreases, $v$ must increase, the kinetic energy of the rotating mass, $\frac{1}{2}mv^2$, increases also, and this increase of kinetic energy corresponds to the work done in opposition to the centrifugal force. Thus the dimensional [illegible] an be justified. [illegible] table collects [illegible] ch have been deduced in reference, first to the fundamental entities of $M$, $L$ and $T$ and also in reference to $M$, $L$, $T$, and $\psi$ or kinematic angle.

Dimensions of Quantities in the Dynamics of Rotation

| *Quantity* | *$M$, $L$ and $T$* | *$M$, $L$, $T$ and $\psi$* |
|---|---|---|
| Angle | Zero dimensions | $\psi$ |
| Angular velocity | $T^{-1}$ | $T^{-1}\psi$ |
| Angular acceleration | $T^{-2}$ | $T^{-2}\psi$ |
| Time | $T$ | $T$ |
| Moment of inertia | $ML^2$ | $ML^2\psi^{-2}$ |
| Angular momentum | $ML^2T^{-1}$ | $ML^2T^{-1}\psi^{-1}$ |
| Moment of momentum | $ML^2T^{-1}$ | $ML^2T^{-1}\psi^{-1}$ |
| Couple or torque | $ML^2T^{-2}$ | $ML^2T^{-2}\psi^{-1}$ |
| Work or energy | $ML^2T^{-2}$ | $ML^2T^{-2}$ |
| Centrifugal force | $ML^2T^{-1}$ | $ML^2T^{-1}\psi$ |

**Moment of Inertia of Geometrical Figures.** A quantity generally termed moment of inertia enters into the calculations of the deflections of beams under various conditions of loading, which is really a geometrical characteristic of the figure of the cross-section of the beam in reference to a stipulated line or axis. This

moment of inertia of a geometrical figure is defined by the formula $\Sigma ar^2$, where $a$ stands for an infinitesimal element of area, and $r$ for its distance from the assigned axis or line. The dimensions of this quantity are $L^2 \times L^2 = L^4$, and it is of a different character from the moment of inertia that determines the kinetic energy of a body rotating with unit angular velocity. The quantity $\Sigma ar^2$ is best termed a second moment, and to convert it to a true moment of inertia it must be multiplied by the mass of unit area, considered constant. This gives $L^4 \times \frac{M}{L^2} = ML^2$, which agrees with the dimensions of moment of inertia in terms of $M$, $L$ and $T$.

To illustrate this point, consider the formula for the deflection $d$ of the free end of a beam of length $l$, loaded with a weight $W$, at the free end. This formula is:

$$d = \frac{1}{3}\frac{Wl^3}{EI},$$

where $E$ is Young's modulus for the material of the beam, and $I$ is the so-called moment of inertia of the figure of section of the beam about an axis through its centre of area. $W$, the weight, has the dimensions of a force or $[F]$. $E$ has the dimensions of stress or $\frac{F}{L^2}$. Thus, substituting dimensions in the formula, and using $ML^2$ for those of $I$, we obtain:

$$L = [F] \times L^3 \times \frac{L^2}{[F]} \times \frac{1}{ML^2} = \frac{L^3}{M},$$

and the dimensions of the two sides of the equation do not agree.

When, however, we use the dimension $L^4$ of a second moment, for $I$, we obtain:

$$L = [F] \times L^3 \times \frac{L^2}{[F]} \times \frac{1}{L^4} = L,$$

which is correct.

## CHAPTER V

## DIMENSIONS OF THERMAL QUANTITIES

**Quantity of Heat.** If a body absorbs heat its temperature rises; if it loses heat, its temperature falls. The [illegible]

posed, called its specific heat, $s$. Otherwise $Q = m\phi s$, where $Q$ is the measure of a quantity of heat in terms of $m$, $\phi$ and $s$, or in what are usually called thermal units The quantity, specific heat, $s$, is defined as the ratio of the heat required to produce unit rise of temperature of the body to that required to produce the same temperature rise of an equal mass of pure water. Accordingly, thermal units are quantities of heat required to produce unit temperature rises in unit masses of water. There are four kinds of thermal units in common use. The mean British Thermal Unit, B.Th.U., is the 1/180th part of the heat required to raise the temperature of 1 pound mass of pure water from the melting point of ice to the boiling point of the water at standard atmospheric pressure. Otherwise, the B Th.U. is approximately defined as the heat required to raise the temperature of a pound of water by 1 degree F. The Centigrade Heat Unit, C.H.U. or pound-calorie, is the 1/100th part of the heat required to raise the temperature of a pound of water from the melting point of ice to the boiling point of water at standard pressure, or approximately the heat required to raise the temperature of a pound of water by 1 degree C. The scientific unit of heat, the calorie, or gram-calorie, is the quantity of heat required to raise the temperature of a gram of pure water from 15 to 16 degrees C. The great calorie, or kilogram-calorie, is the

quantity of heat required to raise the temperature of 1000 grams of water by the same amount.

According to the first law of thermo-dynamics, heat and work are equivalent. A definite quantity of work will produce a corresponding quantity of heat, and a definite quantity of heat can be produced by the dissipation or disappearance of a corresponding amount of mechanical work. Quantity of heat can therefore be measured in dynamical or work units by the amount of work required to produce it. Thus, dynamically, a quantity of heat may be measured in ft.-lbs. in the British Engineers' System of units, in ergs in the C.G.S. system, in joules in the M.K.S. system, or in kilowatt-hours in electrical units.

The measures of a quantity of heat in these two ways are of an essentially different character. Heat is objectively manifested by the fact that its absorption produces temperature rise and consequently actual measures of heat as heat must depend upon a temperature magnitude. The dependence of temperature upon the fundamental dynamical units of mass, length and time is not apparent. Further, measures of heat as heat are stated in terms of a reference substance, water. Measures of heat in thermal units do not therefore correspond to dynamical measures in an absolute system of units. Measures of heat in dynamical units, on the other hand, are really measures of equivalent mechanical work. The connection between the two measures of quantity of heat, or between the various thermal units and dynamical units of work, is given by numerical factors which are known as the mechanical equivalents of heat, and are denoted by the symbol $J$. To each thermal unit of heat there corresponds a separate value of $J$ for each dynamical work unit. Thus a B.Th.U. is equal to so many ft.-lbs., so many ergs and so many joules, and the number of ft.-lbs., ergs or joules is each a value of $J$, the mechanical equivalent of heat.

**Temperature.** Transfer or flow of heat may be conceived to be the result of a temperature difference, so that temperature is a kind of thermo-motive force. Accordingly, temperature difference is a factor that enters into a statement of the magnitude of nearly all thermal quantities. We have already said that, although temperature almost certainly depends in some way upon the fundamental dynamical entities of mass, length and time, the nature of this dependence is not known and it cannot be inferred with certainty. Uncertainty of the dimensions of temperature is a fundamental difficulty in the application of dimensional theory to the science of applied heat, because, *till* the temperature dimensions can be assigned, the dimensions of other thermal quantities cannot be calculated. This difficulty can be overcome, or rather, avoided in two ways. First, dimensions of temperature in terms of

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can be recognised and admitted, and temperature treated as an auxiliary dimensional entity to which a special symbol $[\theta]$ is assigned. By this latter means, the dimensions of thermal quantities are worked out in terms of $M$, $L$, $T$ and $\theta$ We may note here an essential difference between an auxiliary dimension $[\theta]$ for temperature, and the auxiliary dimension $[\psi]$ which was used in Chapter IV for kinematic angle or amount *of turning*. *Kinematic* angle is known with certainty to be independent of mass, length and time, and it is introduced as a fourth dimensional entity into the dimensional theory of rotational dynamics to give this theory greater generality and precision. The use of an auxiliary dimensional symbol $[\theta]$ for temperature does not mean that temperature is independent of $M$, $L$ and $T$, but that, in the present state of our knowledge, we are unable

to express in a dimensional formula the precise manner in which temperature depends upon the fundamental dynamical entities.

Dimensions of Thermal Quantities in terms of $M$, $L$, $T$ and $\theta$. We have seen that the measure of a quantity of heat, $Q$, in thermal units is given by the formula, $Q=m\phi s$, where $m$ is the mass of the body absorbing $Q$, $\phi$ its temperature rise, and $s$ its specific heat. The work equivalent to this quantity of heat, or its measure in work units, $H$, is connected with $Q$ by the formula $H=JQ$, where $J$ is the mechanical equivalent. Specific heat, a specific property of a material substance, is defined as the ratio of two quantities of heat; it is therefore a pure number of zero dimensions. If we substitute dimensions in the equation:

$$H=JQ=J\times m\phi s,$$

we have

$$\frac{ML^2}{T^2}=J\times M\times\theta\times 1,$$

and

$$J=\frac{L^2}{T^2\theta}=L^2T^{-2}\theta^{-1}.$$

The thermal capacity of a body, or the quantity of heat required to raise its temperature by unit amount, is $\dfrac{Q}{\phi}$ in thermal units, and has the dimensions

$$M\theta\div\theta=M.$$

Measured in dynamical units, the thermal capacity has the dimensions $\dfrac{JQ}{\theta}=ML^2T^{-2}\theta^{-1}$ and the specific thermal capacity of a substance, or the heat in dynamical units required to raise the temperature of unit mass by unit amount has the dimensions $L^2T^{-2}\theta^{-1}$, the same as those of $J$.

Latent heat is the quantity of heat required to produce change of state without change of temperature in

unit mass of a substance. Its dimensions are therefore those of $\frac{Q}{m}=\frac{m\phi}{m}$, or $[\theta]$.

The coefficient of expansion of a substance is the fractional change in its length or volume per degree of temperature change A fractional change of length is the ratio
length :
pure nur
coefficient of expansion are therefore those of $\frac{1}{\phi}$ or $[\theta^{-1}]$.

The coefficient of absorption or emission, $A$, of a body having a temperature difference $\phi$ relative to surrounding objects is given by the formula :

$$A=\frac{Q}{ta\phi},$$

where $\frac{Q}{t}$ is the heat gained or lost per second, and $a$ the area of its surface in which heat is absorbed or from which it is emitted Substituting dimensions, we have, as the dimensions of $A$ .

$$[A]=M\theta\times\frac{1}{T}\times\frac{1}{L^2}\times\frac{1}{\theta}=\frac{M}{L^2T}=ML^{-2}T^{-1}.$$

The quantity of heat flowing per second, $\frac{Q}{t}$, through a path of length $l$, and uniform cross-sectional area $a$, between the ends of which a temperature difference $\phi$, is maintained, is given by the equation :

$$\frac{Q}{t}=\frac{ka\phi}{l},$$

where $k$ is the thermal conductivity of the substance of the path. Thus :

$$k=\frac{Ql}{at\phi},$$

and substituting dimensions of the quantities on the right-hand side of the equation, we obtain, as the dimensions of $k$:

$$[k]=M\theta \times L \times \frac{1}{L^2} \times \frac{1}{T} \times \frac{1}{\theta} = \frac{M}{LT} = ML^{-1}T^{-1}.$$

The diffusivity of a substance is defined as the quantity $\frac{k}{C}$, where $k$ is the thermal conductivity and $C$ the heat capacity per unit volume. The dimensions of $C$ are those of $\frac{Q}{V\phi}$, where $V$ stands for volume, or $\frac{M}{L^3}$. Thus the dimensions of diffusivity are:

$$\frac{M}{LT} \div \frac{M}{L^3} = \frac{L^2}{T} = L^2T^{-1}.$$

The thermal resistance of a heat path is the temperature difference of its ends required to produce heat flow at the rate of 1 watt, or 1 joule per second. If $P$ stands for power in watts, thermal resistance is equal to $\frac{\phi}{P}$. As the dimensions of power are $\frac{ML^2}{T^3}$, the dimensions of thermal resistance are:

$$\theta \div \frac{ML^2}{T^3} = \frac{\theta T^3}{ML^2} = M^{-1}L^{-2}T^3\theta.$$

Temperature gradient, the change of temperature per unit length of a path of heat flow has the dimensions, $\theta \div L$ or $L^{-1}\theta$.

The foregoing theory of the dimensions of thermal quantities, based upon the treatment of temperature as an auxiliary dimensional entity, is due to Rucker, and will be found in many textbooks. It avoids any assumptions about the ultimate nature of temperature that are at all questionable, but, as $\theta$ almost certainly depends in some unknown way on $M$, $L$ and $T$, it leaves the dimensional theory of thermal quantities in a somewhat vague and uncertain state. The thoughtful reader

may already have noticed an apparent anomaly which arises from this theory, the assigning of dimensions to the quantity $J$, the mechanical equivalent, which, at first thought seems to be a mere numerical conversion factor, whereby, from the measure of a quantity of heat in terms of one unit, its measure in terms of another unit may be calculated.

Dimensions of Thermal Quantities in terms of $M$, $L$ and $T$. The use of a special dimensional symbol for the dimensions of temperature can be avoided if it is assumed that the quantity $J$ is a dimensionless conversion factor, so that the first law of thermodynamics asserts that a quantity of heat, defined by the formula $m\phi s$, has the same dimensions as those of the quantity of work which will produce it, and to which it is objectively equivalent. As the dimensions of work are $ML^2T^{-2}$, and as $s$, specific heat, is a pure number, this dimensional identity leads to the equation :

$$\frac{ML^2}{T^2}=M\theta \quad \text{or} \quad \theta=\frac{L^2}{T^2}=L^2T^{-2}.$$

Temperature, therefore, has the dimensions of a square of a velocity.

We may here observe that the assignment of the dimensions $L^2T^{-2}$ to temperature can be justified by Joule's kinetic theory of gases, according to which the temperature of a gas is proportional to the mean kinetic energy of a molecule. The temperature of a mass of gas is therefore proportional to the total molecular kinetic energy divided by the mass of a molecule and has the dimensions, energy $\div$ mass, or $\frac{ML^2}{T^2}\div M=\frac{L^2}{T^2}$.

The kinetic theory of gases is supported by much experimental evidence, but it remains a theory and it refers only to one ideal class of substances. The deduction of the dimensions of temperature from this theory can therefore be regarded as no more than an indication that the dimensional formula $L^2T^{-2}$ may

probably be true: it cannot be considered as a strict proof that the formula is correct.

Having assigned the dimensions $L^2T^{-2}$ to temperature, the symbol $[\theta]$ can at once be eliminated in the formulae in the preceding section that contains it, by substituting $L^2T^{-2}$ for $\theta$. Thus $J$ becomes a dimensionless number as does specific thermal capacity. The dimensions of latent heat become $L^2T^{-2}$, and those of coefficient of expansion, or $\theta^{-1}$, $L^{-2}T^2$.

Coefficient of absorption, thermal conductivity and diffusivity all have dimensional formulae independent of $\theta$, and these formulae are therefore unchanged for the $M$, $L$ and $T$ system.

Thermal resistance measured by $\frac{\phi}{P}$ will have the dimensions:

$$\frac{L^2}{T^2} \times \frac{T^3}{ML^2} = \frac{T}{M} = M^{-1}T.$$

The dimensions of temperature gradient will be $\frac{L^2}{T^2} \div L$ or $LT^{-2}$.

Dimensions of Thermal Quantities

| *Quantity* | $M, L, T$ *and* $\theta$ | $M, L$ *and* $T$ |
|---|---|---|
| Quantity of heat | $M\theta$ | $ML^2T^{-2}$ |
| Temperature | $\theta$ | $L^2T^{-2}$ |
| Specific heat | Zero dimensions | Zero dimensions |
| Mechanical equivalent, $J$ | $L^2T^{-2}\theta^{-1}$ | ,, |
| Specific thermal capacity in dynamical units | $L^2T^{-2}\theta^{-1}$ | ,, |
| Latent heat | $\theta$ | $L^2T^{-2}$ |
| Coefficient of expansion | $\theta^{-1}$ | $L^{-2}T^2$ |
| Coefficient of absorption or emission | $ML^{-2}T^{-1}$ | $ML^{-2}T^{-1}$ |
| Thermal conductivity | $ML^{-1}T^{-1}$ | $ML^{-1}T^{-1}$ |
| Diffusivity | $L^2T^{-1}$ | $L^2T^{-1}$ |
| Thermal resistance | $M^{-1}L^{-2}T^3\theta$ | $M^{-1}T$ |
| Temperature gradient | $L^{-1}\theta$ | $LT^{-2}$ |

The Mechanical Equivalent of Heat, $J$. The two sets of dimensional formulae for thermal quantities are, as we have already pointed out, based upon two radically different conceptions of the essential character of the quantity $J$, the mechanical equivalent of heat. The necessity for the use of the auxiliary dimensional symbol $[\theta]$ arises because of the assumption that the measure of a quantity of heat in thermal units has dimensions different from those of the quantity of work to which, according to the first law of thermodynamics, it is exactly equivalent. A dimensional formula in terms of the dynamical entities $L$ and $T$ is assigned to temperature on the assumption that the measures of quantity of heat and equivalent amount of work have the same dimensions. The one assumption assigns a dimensional formula to $J$, the other treats $J$ as a dimensionless number. Each of these ideas raises difficulties. From one point of view $J$ appears plainly to be a mere numerical conversion factor. As volume or capacity can be measured either in cubic inches, cubic feet and so on in one way, or in pints, quarts and so on in another, to each measure in the one kind of unit such as cubic feet, there will be a conversion factor for turning this into the equivalent in pints or quarts, and these conversion factors will all be dimensionless numbers. So, it would appear, to each measure of heat in such thermal units as B.Th.U.'s or calories there will correspond a conversion factor for turning this measure into the equivalent in ft.-lbs. or ergs and so on, and that these factors are dimensionless numbers. On the other hand, the assumption that $J$ is a dimensionless number leads to another difficulty, that the quantity specific thermal capacity, or the heat required to produce unit temperature rise in unit mass of a substance, is a dimensionless number like specific heat defined as the ratio of two quantities of heat.

We can explain and illustrate this difficulty best by an analogy. Consider the equation $m = SV$, where $m$ is

the mass of a body in grams, $S$ its specific gravity, and $V$ its volume in cubic centimetres. As specific gravity is the ratio of two masses, it is of zero dimensions, so that we have the dimensional formula $M=L^3$. This dimensional formula will not fit an absolute system of units in which the mass and length units are arbitrarily defined. In fact, the formula implies that the mass unit is defined in terms of that of unit volume of a standard substance, water. Thus $M=L^3$ is a dimensional equation which is not universally true. The general relation between $m$ and $V$, $m=\sigma V$, is true only if $\sigma$ is, not specific gravity, but density, defined as the mass of unit volume of the substance concerned, and density is of course a dimensional quantity. The equation $m=SV$ to be universally true must be modified by the introduction of another factor and be stated $m=SV\sigma_1$, where $\sigma_1$ is the mass of unit volume of water in the system of units that is relevant. $\sigma_1$ is essentially a density of dimensions $\frac{M}{L^3}$ so that the equation $m=SV\sigma_1$ is dimensionally equivalent to $M=M$.

If the reader has followed and carefully considered the foregoing argument, he will probably be in a position to understand the doubt of the validity of assuming that $J$, the mechanical equivalent of heat, is a dimensionless conversion factor. The measure of a quantity of heat $Q$, in units based upon a unit of mass and one of temperature difference, as $m\phi s$, where $s$ is a dimensionless ratio, is true only if the heat unit is defined in terms of unit $m$, unit $\phi$, and in reference to the standard substance, water, which is used to define $s$. As the formula, $m=SV$ for mass in terms of specific gravity and volume is true only for a system of units in which the mass of unit volume of water is unity, so the equation $Q=m\phi s$ is true only for heat units in which the quantity of heat to raise the temperature of unit mass of water by unit amount is also unity. Further,

as to make the equation $m=SV$ true for any absolute system of units, the factor $\sigma_1$, the density of water must be introduced, giving $m=SV\sigma_1$, where $\sigma_1$ is a dimensional quantity, so, to make the equation $Q=m\phi s$ true for a system of absolute units based upon arbitrary units of mass, length and time, an additional factor $J$ must be introduced giving $Q=Jm\phi s$, where $J$ is the measure of heat in dynamical units required to produce unit rise of temperature in unit mass of water, and is an essentially dimensional number like $\sigma_1$ the density of water. In other words, $J$, the mechanical equivalent of heat, can also be considered as a physical characteristic of water, its specific thermal capacity in dynamical units.

If the foregoing views are correct, then it seems that the theory of the dimensions of thermal quantities based upon the assumption that $J$ is a pure number, is fundamentally unsound, and that, notwithstanding the apparent sanction given by the kinetic theory of gases, the validity of the dimensional formula $L^2T^{-2}$ for temperature is open to serious doubt. If the reader pursues his studies of applied physics, he will find that the dimensions of thermal quantities are given in both of the above ways, that is, in terms of $M$, $L$, $T$ and $\theta$, and in terms of $M$, $L$ and $T$ only, and the latter system of thermal dimensions has the authoritative support of the late Dr. Lanchester in his treatise 'Theory of Dimensions'. Both sets of statements of thermal dimensions lack the clarity and coherence of the dimensional theory of dynamical quantities, the relation of which to the fundamentals $M$, $L$ and $T$ can be worked out with certainty. This lack of clarity is, perhaps, a necessary consequence of the fact that a temperature unit has practically to be defined in reference to a physical property of a material body, water, or a gas that is supposed to approximate in fundamental character to a so-called perfect gas. Although there is a thermodynamic scale of temperature, it is not at the present

time possible exactly to define a temperature unit in terms of dynamical units. Till this is possible, it would seem desirable to avoid the ambiguity and weakness of treating $J$ as dimensionless, and to retain the auxiliary dimensional symbol $[\theta]$ in the dimensional formulae for thermal quantities.

The following table gives the various mechanical equivalents of heat, that is, the values of each thermal unit in terms of the three dynamical units corresponding to the British Engineers', the C.G.S. and the M.K.S. systems. $J$ is an experimentally determined number, and at the present time its value is not known quite as closely as 1 part in 10,000. We shall see, in a subsequent section, how, given one value of $J$, others are calculated.

Mechanical Equivalent of Heat

| Thermal Unit | Dynamical Unit | | |
|---|---|---|---|
| | *ft.-lbs.* | *ergs* | *joules or watt-seconds* |
| B.Th.U. | 777·8 | $1{\cdot}07 \times 10^{10}$ | 1070 |
| C.H.U. | 1400 | $1{\cdot}93 \times 10^{10}$ | 1930 |
| gram-calorie | 3·09 | $4{\cdot}184 \times 10^{7}$ | 4·184 |
| kilogram calorie | 3090 | $4{\cdot}184 \times 10^{10}$ | 4184 |

**The Gas Constant.** The relation between the pressure $p$, the volume $V$, and the absolute temperature $\phi_a$, of a so-called perfect gas is expressed by the equation,

$$pV = mR\phi_a \quad \text{or} \quad \frac{p}{\sigma} = R\phi_a,$$

where $m$ is the mass of the gas, $\sigma$ its variable density, and $R$ is a constant for the particular gas. These

equations are very approximately true for actual gases at temperatures considerably above the critical temperature. $R$ is often called the gas constant, and it is equal to $\frac{pV}{m\phi_a}$.

If we substitute the dimensions of the factors in this expression, then, as the dimensions of pressure are those of force $\div$ area we have for the dimensions of $R$:

$$[R]=\frac{ML}{T^2}\times\frac{1}{L^2}\times L^3\times\frac{1}{M}\times\frac{1}{\theta}=L^2T^{-2}\theta^{-1},$$

and these dimensions are the same as those of $J$. We observe that the quantity $pV$ has the dimensions $ML^2T^{-2}$, those of work, so that $R$, objectively, is work per unit mass per degree of temperature. $R$ divided by $J$ or $\frac{R}{J}$ gives the thermal equivalent of work, or thermal units per unit of mass per degree of temperature. $\frac{R}{J}$ is of zero dimensions, and it has the character of a specific heat. Actually, $\frac{R}{J}$ is the difference of the specific heats of a perfect gas at constant pressure and constant volume, the greater value of the former specific heat being due to the external work done in the expansion of the perfect gas, the internal work being zero

If $V_0$ stands for the volume of a mass $m$ of gas at standard pressure $p_0$ and standard absolute temperature $\phi_0$, then $R=\frac{p_0V_0}{m\phi_0}$, where $p_0$, $V_0$ and $m$ and $\phi_0$ are all known quantities. Thus $R$ can be calculated for any particular gas. If the mass $m$ of the gas is what is called the gram-molecule, or that of a standard volume of 22,320 cubic centimetres at standard atmospheric pressure, and at a temperature of 273 degrees absolute, the quantity $\frac{V_0}{m}$ is the same for all gases, so that the gas

constant $R$ defined in this way is independent of the particular gas, and has the invariable value of $8{\cdot}14 \times 10^7$ ergs per degree. This special value of $R$, divided by $J$, gives an equivalent in thermal units, which is approximately 2 gram-calories per degree. Thus, for all gases which approximate in characteristics to those of a perfect gas, $pV = R\phi_a$, where $R$ is equal to $8{\cdot}14 \times 10^7$ ergs per degree, and $\frac{R}{J}$ is 2 gram-calories per degree.

The several meanings of the gas constant must be carefully distinguished. $R$ in the equation $\frac{p}{\sigma} = R\phi_a$ is expressed in work units per unit mass per degree and depends upon the particular gas, $\frac{R}{J}$ is expressed in thermal units per unit mass per degree. If the equation $pV = R\phi_a$ refers to a gram-molecule of the gas and not to unit mass, $R$ in work units per degree, or $\frac{R}{J}$ in thermal units per degree, is independent of the gas, provided that the temperature remains considerably above the critical value.

Conversion Factors for Thermal Quantities. We have, in a previous section, discussed the merits of the two systems of dimensional formulae for thermal quantities, from the purely theoretical point of view. When we come to the practical use of these formulae in the calculation of conversion factors, we find that the system in which a dimensional symbol is retained for temperature has material advantages. We shall illustrate the use of thermal dimensions formulae in this way by a few examples.

What is the factor for converting B.Th.U. to gram-calories? Here the units are those of a quantity of heat $Q$ with the dimensions $M\theta$. The basic factor for $M$ is the number of grams to one pound of mass or

452, and the number of Centigrade degrees to one Fahrenheit degree, or $\frac{5}{9}$. Thus :

$$1 \text{ B.Th.U.} = 454 \times \tfrac{5}{9} = 252 \text{ gram-calories.}$$

If the value of $J$, the mechanical equivalent of heat, is 778 ft.-lbs. per B.Th.U., what is its value in ergs per gram-calorie? Here we must note that in using dimensional formulae, the units in each statement of $J$ must be consistent. that is, each must belong to the same absolute dynamical system. Now the B Th.U. is not consistent in this way ; the work unit, the ft.-lb. corresponds to the British Engineers' System, while the mass unit, the pound, corresponds to the F.P.S. system We must therefore change the given value of $J$ from ft.-lbs. per pound per degree F. to ft.-lbs. per slug per degree F. This value of $J$, consistent with the British Engineers' System of units, is therefore $778 \times 32{\cdot}2$, because 1 slug of mass $= 32{\cdot}2$ pounds.

*The dimensional formula for $J$ is $\frac{L^2}{T^2\theta}$. As the unit* of time is the same for both the British Engineers' and the C.G.S. systems of units, the required conversion factor depends upon $\frac{L^2}{\theta}$, and it is the square of the centimetres in 1 ft. divided by the C degrees corresponding to 1 degree F. These two basic conversion factors are respectively 929 and $\frac{5}{9}$. Thus :

| | |
|---|---|
| $J = 778 \times 32{\cdot}2 \times 929 \times \frac{9}{5}$ | ergs per gram-calorie |
| $\quad = 7{\cdot}78 \times 3{\cdot}22 \times 9{\cdot}29 \times 1{\cdot}8 \times 10^5$ | do. |
| $\quad = 419 \times 10^5$ | do. |
| $\quad = 4{\cdot}19 \times 10^7$ | do. |

Thus, alsc

1 joule, or 1

To conver

C.H.U., the basic conversion factor is merely the

degrees C. in a degree F. or $\frac{5}{9}$, so that:

$$J = 778 \div \tfrac{5}{9} = 1400 \text{ ft.-lbs. per C.H.U.}$$

The thermal conductivity of slate is quoted as 1 B.Th.U. per hour, per foot, per degree F.; what is its value in gram-calories per second, per centimetre, per degree C.? Here the dimensional formula for $k$, thermal conductivity, is $\frac{M}{LT}$ and the conversion factor is $\frac{n_1}{n_2 n_3}$, where $n_1$ is the number of grams in 1 pound, 453; $n_2$ the centimetres in 1 foot, 30·48; and $n_3$ the seconds in 1 hour, 3600. The conversion factor is therefore:

$$\frac{n_1}{n_2 n_3} = \frac{453}{30{\cdot}48 \times 3600}.$$

The required value of $k$, the thermal conductivity in C.G.S. units, is therefore:

$$1 \times \frac{453}{30{\cdot}48 \times 3{\cdot}6} \times 10^{-3} \text{ gram-calories per second per cm. per degree C.}$$

$= 4{\cdot}14 \times 10^{-3}$ do.

or, approx. 0·004 do.

Example of the Application of Thermal Dimensional Formulae. A horizontal wire heated by the passage of an electric current and placed in a moving cooling medium will lose heat at a rate proportional to its length $l$, and to its temperature difference $\phi$ relative to that of the medium. This rate, $\frac{Q}{tl\phi}$ per second per unit length per degree, depends also in some way on the velocity $v$ of the medium, upon the product $k \times C$ where $k$ is its thermal conductivity and $C$ its thermal capacity per unit volume and upon the diameter $d$ of the wire. Symbolically:

$$\frac{Q}{tl\phi} = b \times v^x \times (kC)^y \times d^z,$$

where $b$ is a dimensionless constant. The dimensions

of $C$ are those of $Q$ per degree per unit of volume or $\frac{M\theta}{\theta L^3} = \frac{M}{L^3}$. Substituting the dimensions of other quantities, we have, as the dimensions of $(kC)$ are $\frac{M}{LT} \times \frac{M}{L^3} = \frac{M^2}{L^4 T}$:

$$\left[\frac{Q}{tl\phi}\right] = \frac{M\theta}{TL\theta} = ML^{-1}T^{-1} = [LT^{-1}]^x \times [M^2L^{-4}T^{-1}]^y \times [L]^z,$$

and, equating dimensions of $M$, $L$ and $T$ in the usual way, we obtain:

$$1 = 2y,$$
$$-1 = x - 4y + z,$$
$$-1 = -x - y.$$

From the first and third equations we have $y = \frac{1}{2}$ and $x = \frac{1}{2}$, and substituting these values in the second equation:

$$1 = 2 - \tfrac{1}{2} - z \quad \text{or} \quad z = \tfrac{1}{2},$$

so that $\frac{Q}{tl\phi} = b \times \sqrt{(vkCd)}$.

*If the temperature difference $\phi$ is constant, the rates*

per unit length which is proportional to $\frac{1}{d^2}$. Thus, for an assigned steady temperature rise,

$$\frac{I^2}{d^2} \textit{ is proportional to } \sqrt{d},$$

*and $I$, the current required to produce this temperature rise, is proportional to $\sqrt{(d^2 \times \sqrt{d})} = d^{\frac{5}{4}}$, for fixed values of $v$, $k$ and $C$ applicable to the cooling medium.* This shows that if the fuse-element wire of a fuse is cooled *by convection only, the current required to raise the temperature of the fuse wire to melting point is proportional to the $\frac{5}{4}$ power of its diameter.*

# CHAPTER VI

## DIMENSIONS OF ELECTRICAL QUANTITIES

**Electrical Units.** The applications of dimensional theory to the science of applied electricity raises difficulties of a more serious character than those so far encountered. Hitherto, it has been found desirable to introduce into dimensional analysis a symbol for a dimensional entity additional to those for mass, length and time, for two reasons; the first, to give greater generality to the dimensional formulae for the quantities applicable to the dynamics of rotation and, the second, to avoid an insufficiently justified assumption of the dimensions of temperature in terms of $M$, $L$ and $T$. There was no essential need for these additional symbols; the dimensional formulae of the dynamics of rotation are customarily expressed in terms of $M$, $L$ and $T$ only, while the ascription of the dimensions $L^2T^{-2}$ for temperature has authoritative sanction. When, however, we come to work out the dimensions of electrical quantities, the use of a fourth dimensional symbol is found to be absolutely necessary in order to avoid a contradiction of the fundamental axiom or principle, that the dimensions of a physical quantity can be expressed in one way only. The question in the application of dimensional theory to applied electricity is not whether an additional symbol is desirable, but what quantity is best chosen as a fourth dimensional entity.

There are two distinct absolute systems of electrical units which are based upon the fundamental C.G.S. units of length, mass and time. The C.G.S. Electrostatic system depends upon the idea of an isolated quantity of electricity on the surface of an insulated body. The definition of electrical units in this system is based upon the law that two identical quantities of

electricity, $q$, supposed to be concentrated at points at a distance $r$, in a medium of permittivity $\kappa$, will be acted upon by a force that is proportional to $\frac{q^2}{r^2\kappa}$. A quantity of electricity, or a charge as it is sometimes termed, on the surface of a body, tends to escape to earth, and this tendency is called the potential $V$. The product $qV$ is proportional to the energy in the charge. The permittivity of a medium, $\kappa$, is usually defined as the ratio of the quantity on a definite body for unit potential in this medium, to the quantity for unit potential in a vacuum or in free space The permittivity of free space is therefore unity. Unit quantity of electricity in the C.G.S. Electrostatic system is defined as that which will act upon a similar quantity at one centimetre distance in free space with a force of 1 dyne. Unit potential associated with unit quantity corresponds to 1 erg of energy. Unit current is the flow of unit quantity in unit time, or one second

The second absolute system of electrical units is

upon by a force proportional to $\frac{m^2}{r^2\mu}$, and unit magnetic pole is defined as that which will act upon a similar magnetic pole at 1 centimetre distance in free space with a force of 1 dyne. The property, permeability, of a medium is based upon the conception that a magnetic pole is the origin of lines of magnetic flux which arise from a magnetising force. The permeability $\mu$ of a medium is defined as the ratio of the magnetic flux produced by a stipulated magnetising force in a stipulated path in the medium, to the flux produced by the same magnetising force in the same path in free space. The relation of electric current to magnetic pole

strength depends upon the law that if a current $I$ flows in a circular path of length $l$ and radius $r$, a magnetic pole of strength $m$ at the centre of the path will be acted upon by a force proportional to $\frac{mIl}{r^2}$, and this force is independent of the medium in which the action takes place. Unit current in the C.G.S. Electromagnetic system is defined as that which, flowing in a circular path 1 centimetre in length and 1 centimetre in radius, will act on unit magnetic pole at the centre of the path with a force of 1 dyne. Unit potential difference causing the flow of unit current will correspond to power at the rate of 1 erg per second. The connection of electric current with the basic magnitude, unit pole, in the Electromagnetic system is much more remote than the corresponding connection of current and the basic magnitude, quantity or charge, in the Electrostatic system.

We have, thus, two distinct ways in which a unit of electric current or rate of flow of electricity can be defined in reference to the fundamental units, the centimetre, the gram and the second, and an objective current flowing in a conductor can be measured by either of these units. There is no essential reason why these two measures should be the same, or, in other words, why unit current in the one system should be identical with unit current in the other. But although the numerical measures of current in the two systems need not agree, the dimensions of the magnitude, current, deduced from the physical laws upon which the definitions of the units are based, must be the same because, according to the fundamental axiom of dimensional analysis, the dimensions of the same physical quantity can be expressed in one way and only one way.

Dimensions of Electrical Quantities. We have seen that electrical units, considered as derived units of the C.G.S. system make contact with the simple dynamical

units in two ways; first, in the specification of unit force of 1 dyne for the mutual action of unit quantities, unit magnetic poles, and unit current on unit magnetic pole, and secondly, in the principle that the association of unit quantity of electricity with unit potential difference corresponds to unit quantity of energy or 1 erg. As the quantities $\kappa$ and $\mu$ in the formulae for the [illegible]

flux in the case of [illegible] it appears at first sight that these quantities, [illegible] reference t[illegible] similar in character to specific gravity and specific heat which refer to a reference substance, water, so that $\kappa$ and $\mu$ are magnitudes of zero dimensions. On this assumption we can easily work out the dimensions of current in two ways, the first through the law of the Electrostatic system of units, and the second through the laws of the Electromagnetic system.

If we start with the fundamental law of the Electrostatic system, we have for the force $F$ acting on two quantities, each $q$ in magnitude:

$$F = \frac{q^2}{r^2 \kappa} \quad \text{or} \quad q^2 = F r^2 \kappa,$$

substituting the dimensions of $F$ and $r^2$, and assuming that $\kappa$ is dimensionless:

$$[q^2] = \frac{ML}{T^2} \times L^2 = \frac{ML^3}{T^2}$$

and $$[q] = M^{1/2} L^{3/2} T^{-1}.$$

As current $I$ is measured as $q$ divided by time, the dimensions of current are:

$$[I] = \frac{[q]}{[T]} = M^{1/2} L^{3/2} T^{-2}.$$

If we start with the fundamental law of the Electro-

magnetic system of units, we have for the force $F$ acting on two magnetic poles, each of strength $m$:

$$F=\frac{m^2}{r^2\mu} \quad \text{or} \quad m^2=Fr^2\mu,$$

and, assuming that $\mu$ is dimensionless, we have for the dimensions of $m^2$, or $[m^2]$:

$$[m^2]=\frac{ML}{T^2}\times L^2=\frac{ML^3}{T^2},$$

and $$[m]=M^{1/2}L^{3/2}T^{-1}.$$

The law connecting pole strength $m$ with current $I$ is:

$$F=\frac{mIl}{r^2} \quad \text{or} \quad I=\frac{Fr^2}{ml}.$$

The dimensions of current $[I]$ are therefore given by:

$$[I]=\frac{ML}{T^2}\times L^2\times\frac{1}{L}\times\frac{1}{[m]}$$
$$=\frac{ML^2}{T^2}\times\frac{1}{M^{1/2}L^{3/2}T^{-1}}=M^{1/2}L^{1/2}T^{-1}.$$

By following two separate lines of argument, we have arrived at two dimensional formulae for electric current $I$; the formula $M^{1/2}L^{3/2}T^{-2}$ corresponds to the Electrostatic system of units and the formula $M^{1/2}L^{1/2}T^{-1}$ to the Electromagnetic system. These two formulae are not in agreement as they should be according to the basic axiom of the theory of dimensions. There must, therefore, be some fallacy or false assumption in at least one of the lines of argument. A possible reason for the inconsistency of the two dimensional formulae is that one of the quantitative laws used to establish them is not true, but as the laws of electrical science have long acquired the final sanction of agreeing with experience and experiment we may reject this supposition at once. The only other explanation of the dimensional inconsistency is that the assumption that

… bility $\mu$, are … mensionless, … To obtain a correct formula for the dimensions of an electrical magnitude such as current, we must retrace our steps and use an additional symbol for the unknown dimensions of either $\kappa$ or $\mu$.

If we … as define … assign tl … permittivity, we have, from the basic equation $q^2 = Fr^2\kappa$, the dimensional equations:

$$[q^2] = \frac{ML}{T^2} \times L^2 \times \kappa,$$

and $$[q] = M^{1/2}L^{3/2}T^{-1}\kappa^{1/2},$$

and the dimensions of current, being those of $[q] \div [T]$, are given by:

$$[I] = M^{1/2}L^{3/2}T^{-2}\kappa^{1/2}.$$

Similarly, by introducing a dimensional symbol $[\mu]$, for permeability in the fundamental law of the Electromagnetic system of units we find that the dimensions of magnetic pole strength are $M^{1/2}L^{3/2}T^{-1}\mu^{1/2}$, and from the law connecting current $I$ and pole strength $m$, $F = \frac{mIl}{r^2}$ or $I = \frac{Fr^2}{ml}$, we obtain the dimensional equation:

$$[I] = \frac{ML}{T^2} \times L^2 \times \frac{1}{L} \times \frac{1}{[m]}$$

$$= \frac{ML^2}{T^2} \times \frac{1}{M^{1/2}L^{3/2}T^{-1}\mu^{1/2}} = M^{1/2}L^{1/2}T^{-1}\mu^{-1/2}.$$

We have now obtained two distinct dimensional formulae for the same physical magnitude, current, and these two formulae must be equivalent in that they both mean the same thing. We can therefore equate them and obtain:

$$M^{1/2}L^{3/2}T^{-2}\kappa^{1/2} = M^{1/2}L^{1/2}T^{-1}\mu^{-1/2},$$

or
$$\kappa^{1/2}\mu^{1/2} = L^{-1}T,$$

or
$$\frac{1}{\kappa^{1/2}\mu^{1/2}} = LT^{-1}.$$

The magnitude $\frac{1}{\kappa^{1/2}\mu^{1/2}}$ where $\kappa$ and $\mu$ stand respectively for the measures of the permittivity and the permeability of a medium, has the dimensions of a velocity. $\kappa$ and $\mu$ cannot, therefore, both be dimensionless. We can express the dimensions of either $\kappa$ or $\mu$ in terms of the other, and $L$ and $T$ by the equations :

$$[\kappa] = L^{-2}T^{2}\mu^{-1},$$
$$[\mu] = L^{-2}T^{2}\kappa^{-1}.$$

We can proceed with the calculation of the dimensions of electrical quantities by using the relation that quantity of electricity $q$, or $It$, multiplied by potential or potential difference $V$, represents energy. Thus, using $[\kappa]$ as a dimensional symbol, we have :

$$[q] \times [V] = ML^{2}T^{-2},$$

and
$$[V] = ML^{2}T^{-2} \times [q]^{-1} = ML^{2}T^{-2} \times M^{-1/2}L^{-3/2}T\kappa^{-1/2}$$
$$= M^{1/2}L^{1/2}T^{-1}\kappa^{-1/2}.$$

And as electrical resistance is defined by $V \div I$, the dimensions of resistance $[R]$ are given by :

$$[R] = [V] \times [I]^{-1}$$
$$= M^{1/2}L^{1/2}T^{-1}\kappa^{-1/2} \times M^{-1/2}L^{-3/2}T^{2}\kappa^{-1/2} = L^{-1}T\kappa^{-1}.$$

Using $[\mu]$ as the additional symbol and proceeding from the dimensions of $I$ as defined by the laws of the Electromagnetic system of units, we have :

$$[V] = ML^{2}T^{-2} \times [IT]^{-1} = ML^{2}T^{-2} \times M^{-1/2}L^{-1/2}\mu^{1/2}$$
$$= M^{1/2}L^{3/2}T^{-2}\mu^{1/2},$$

and
$$[R] = [V] \div [I],$$
$$= M^{1/2}L^{3/2}T^{-2}\mu^{1/2} \times M^{-1/2}L^{-1/2}T\mu^{1/2},$$
$$= LT^{-1}\mu.$$

We shall see later how, having established dimensional formulae for $I$ and $V$, the dimensions of all other electrical and electromagnetic magnitudes arising in electrical technology can be deduced.

The foregoing arguments have established two facts regarding the theory of the dimensions of electrical quantities. First, the assumption that the quantities, permittivity and permeability are dimensionless like specific gravity and specific heat, is unwarranted. One at least of these quantities must be dimensional. Secondly, one additional dimensional symbol is sufficient to express the dimensions of electrical magnitudes. This latter conclusion follows it should be noted, from two principles or postulates, first that the dimensions of quantity of electricity multiplied by potential difference are those of energy, and, secondly that the force exerted by a current flowing in a circular path on a magnetic pole is independent of the medium in which the action takes place

The dimensions of other electrical magnitudes in terms of $M$, $L$, $T$ and $\kappa$ and in terms of $M$, $L$, $T$ and $\mu$ will be found on the table on page 101. The reader will have noticed that, in this chapter, we have, for the first time, had to use fractional exponents or indices for $M$, $L$ and $T$ in dimensional formulae.

**Ratio of the Magnitudes of Electrical Units.** Owing to the different ways in which an electrical unit such as quantity is defined in the two systems of units, the Electrostatic and the Electromagnetic, the measures of a quantity or charge in terms of the two units are very different numerically. The Electromagnetic unit of quantity is equal to 10 coulombs or 10 ampere-seconds, and this is enormously greater than the Electrostatic unit charge which would act on a similar charge at 1 centimetre distance with a force of 1 dyne, and which is about equal to the charge communicated to an isolated sphere, 1 centimetre radius by a potential of 300 volts.

But why, it may be asked, are corresponding units of the two systems different in magnitude, since each is referred ultimately to the same fundamental units of mass, length and time? The answer to this question is that, in defining the two sets of units, it has been assumed that the permittivity $\kappa$ and the permeability $\mu$ of free space each have the value unity. But as corresponding units do not agree in magnitude, this assumption must be wrong and the $\kappa$ and $\mu$ of free space cannot each be equal to 1; if we assume unity value for one of these quantities, the other must have a value different from unity.

We have seen in Chapter III that the number of derived units of one system that is equal to one corresponding derived unit of another system depends both on the dimensions of the unit, and upon the ratios of the magnitudes of the fundamental units of the two systems. Now, the fundamental units of mass, length and time in the Electrostatic and Electromagnetic systems of units are identical; each is a C.G.S. system. The ratios of the magnitudes of corresponding units of the two systems must therefore depend in some way on the ratios of the measures of permittivity or permeability in the two systems. In the Electromagnetic system the permeability $\mu$ of free space is unity, so that the ratios of corresponding units of the two systems will depend in some way upon the measure of the permittivity of free space in the Electromagnetic system of units. At present, however, there is no means whereby this measure of permittivity can be made directly.

We shall see later, however, how the ratios of the magnitudes of other corresponding units in the two systems can be measured experimentally. Suppose that it has been found that 1 Electromagnetic unit of quantity is equal to $c$ Electrostatic units. Now, according to the basic law defining the Electrostatic system, if two such quantities, each of $c$ Electrostatic units could be concentrated at points 1 centimetre

distance in free space, they would be acted on by a force equal to $\frac{c^2}{r^2\kappa}=\frac{c^2}{1\times 1}=c^2$ dynes. If this law is to fit the Electromagnetic system of units, so that unit quantities at unit distance are acted on by unit force in a medium for which $\kappa=1$, then for a medium, free space, in which the force is $c^2$ dynes, we must have the relation $c^2$ dynes $=\frac{1\times 1}{1\times\kappa}$ where $\kappa$ is the measure of the permittivity of free space in Electromagnetic units. Thus, $\kappa$ of free space in the Electromagnetic system of units must have the value $\frac{1}{c^2}$, and as the permeability $\mu$ of free space in this system is 1, the quantity $\frac{1}{\kappa\mu}$ must be equal to $c^2$.

A comparison of the magnitudes of corresponding [illegible]

receive a charge of electricity is defined as $\frac{q}{V}$, where $q$ is the quantity in the charge and $V$ the potential. Now, the dimensions of capacitance, $C$, are those of $q-V$ or $M^{1/2}L^{3/2}T^{-1}\kappa^{1/2}-M^{1/2}L^{1/2}T^{-1}\kappa^{-1/2}=[L\kappa]$ This shows that capacitance depends on linear dimensions and permittivity only, and, as a matter of fact, the capacitance of a body of suitable shape can be calculated precisely in Electrostatic units from its dimensions and from the value of the permittivity of the adjacent medium assessed on the basis that the permittivity of free space is unity. Further, the magnitude $\frac{q}{V}=\frac{It}{V}$ has the dimensions of $\frac{t}{R}$, for resistance is equal to $\frac{V}{I}$. $C$, therefore, has the same dimensions as $\frac{1}{Rf}$, wh[illegible] is a frequency. Now, the Electromagnetic me[illegible] timetre [illegible]esistance can be made

precisely, and Maxwell devised an experiment in which the Electromagnetic measure of a capacitance $C$ was found in terms of known resistances and the frequency of an oscillating contact maker. From this measure and the Electrostatic measure based upon dimensions and capacitance, Maxwell found:

$$\frac{\text{Electrostatic measure of a capacitance}}{\text{Electromagnetic measure of same capacitance}}=9\times 10^{20}.$$

Now, the dimensions of capacitance are $[L\kappa]$ and, as explained in Chapter III, the number of Electrostatic capacitance units in one Electromagnetic unit must be $n_1 n_2$ where $n_1$ is the ratio of the length units and $n_2$ that of the permittivity units of the two systems. But $n_1=1$ because the centimetre is the length unit for both systems. The Electrostatic capacitance units corresponding to one Electromagnetic unit must be equal to $n_2$, where $n_2$ is the ratio of the Electrostatic unit of permittivity to the Electromagnetic unit. Thus $n_2$ must be equal to $\frac{1}{\kappa}$ where the 1 of the fraction is the Electrostatic measure of the permittivity of free space, and $\kappa$ is the Electromagnetic measure. Thus:

$$n_2=9\times 10^{20}=\frac{1}{\kappa}=\frac{1}{\kappa\mu},$$

and

$$\kappa=\frac{1}{9\times 10^{20}},$$

where $\kappa$ and $\mu$ stand respectively for the electromagnetic measures of the permittivity and the permeability of free space, since the $\mu$ of free space has the value unity.

The permittivity of free space therefore has the numerical value $1/(9\times 10^{20})$ Electromagnetic units, so that, if it were possible for two Electromagnetic unit quantities to be concentrated at points at 1 centimetre

distance in a vacuum, they would be acted upon by a force of $9 \times 10^{20}$ dynes.

We have seen that the quantity $\frac{1}{\kappa\mu}$ has the dimensions $L^2T^{-2}$ or those of the square of a velocity. According to theory, the velocity of propagation in centimetres per second of electromagnetic waves in free space is equal to $\frac{1}{\sqrt{(\kappa\mu)}}$ where $\kappa$ and $\mu$ are the measures of the permittivity and permeability in any C.G.S. system of units. The velocity corresponding to the quantity $\frac{1}{\sqrt{(\kappa\mu)}}$ in Electromagnetic measure is therefore $3 \times 10^{10}$ centimetres per second, and this agrees with the velocity of light found by other methods.

**Practical Electrical Units.** The C.G.S. electrical units of both of the systems so far considered are practically inconvenient for two reasons. First, the magnitudes of these units are unsuitable, and, secondly, the accurate measurement of an electrical magnitude in terms of fundamental units is very difficult. For these reasons, practical electrical units are defined objectively in terms of subsidiary fundamental units of current and resistance which, by exhaustive experiment, have been proved to be highly approximate decimal multiples of the Electromagnetic units.

The practical unit of current, very nearly equal to $\frac{1}{10}$th of a C.G.S. Electromagnetic unit, is defined as the unvarying current which, when passed through a solution of silver nitrate in water in accordance with standard specifications, deposits silver at the rate of 0·001118 grams per second. This unit is known as the 'international ampere'.

The practical unit of resistance, the 'international ohm', which is very nearly equal to $10^9$ C.G.S. Electromagnetic resistance units, is the resistance to an unvarying current offered by a column of mercury at

the temperature of melting ice, 14·4521 grams in mass, of a constant cross-sectional area, and of the length 106·3 centimetres.

From these concretely defined units, other derived electrical units follow. Unit potential difference is given at the ends of unit resistance with an unvarying unit current. Unit potential difference causing the flow of unit current corresponds to 1 watt of power, or 1 joule $=10^7$ ergs per second.

The only Electrostatic unit employed in practice is in the calculation of capacitance from linear dimensions. The result of such a calculation gives a measure in Electrostatic units. To convert this measure to C.G.S. Electromagnetic units, we have seen that it must be divided by $9\times10^{20}$. But as the practical unit of potential difference, the volt, is equal to $\frac{1}{10}\times10^9$ where $\frac{1}{10}$ and $10^9$ are respectively the C.G.S. measures of the ampere and the ohm, the volt $=10^8$ Electromagnetic units. The practical unit of capacitance, the farad, corresponding to a charge of one ampere second per volt is equal to $10\times10^8=10^9$ Electromagnetic units. Thus the farad measure of a capacitance is equal to its measure in Electrostatic units multiplied by

$$10^9\div(9\times10^{20})=\frac{1}{9\times10^{11}},$$

and micro-farads = Electrostatic units multiplied by $\frac{1}{9\times10^5}$.

M.K.S. Absolute Electromagnetic Units. As the energy unit corresponding to the practical electrical units, the ampere and the volt, is the watt-second, joule, or the newton-metre, these electrical units correspond to an absolute system in which the fundamental units of length and mass are the metre and the kilogram respectively. Unit magnetic pole in this M.K.S. Electromagnetic system will be defined as that which acts on a similar pole at 1 metre distance with a

unit force of 1 newton in a medium of permeability equal to unity.

From the fundamental equation connecting current $I$ with pole strength $m$, or $F=\dfrac{Ilm}{r^2}$, we have $m=\dfrac{Fr^2}{Il}$, so that the dimensions of $m$ in terms of $M$, $L$, $T$ and current, are :

$$\frac{ML}{T^2}\times L^2\times\frac{1}{L}\times\frac{1}{[\text{Current}]}=\frac{ML^2}{T^2}\times\frac{1}{[\text{Current}]}.$$

We therefore have :

$$1 \text{ unit M.K.S. magnetic pole}=\frac{n_1 n_2^2}{n_3^2 n_4} \text{ C G S unit poles,}$$

where $n_1$ is the number of grams in one kilogram, 1000 ; $n_2$ the number of centimetres in one metre, 100 ; $n_3=1$ because the time units in the two systems are the same, and $n_4$ is the number of C.G.S. current units in one ampere, $\frac{1}{10}$ Thus :

$$1 \text{ unit M.K.S. pole}=\frac{1000\times 100^2}{1\times\frac{1}{10}}=10^8 \text{ C.G.S. unit poles.}$$

Now, two equal magnetic poles of strength $10^8$ C.G.S. Electromagnetic units at 1 metre or 100 centimetres distance in free space, will be acted upon by a force of :

$$\frac{10^8\times 10^8}{100^2}=10^{12}\text{ dynes}=10^7 \text{ newtons,}$$

as the permeability of free space in C G S. Electromagnetic units is 1. In M.K.S. units, these two unit poles at unit distance of 1 metre will be acted upon by a force of $10^7$ newtons in a medium of permeability $\mu_1$ where $\mu_1$ is the M K S. measure of the permeability of free space ; therefore

$$\frac{1\times 1}{1^2\times\mu_1}=10^7 \quad\text{or}\quad \mu_1=10^{-7}.$$

$10^{-7}$ is therefore the measure of the permeability of free space in M.K.S. units.

**The Fourth Dimensional Entity for Electrical Quantities.** We have seen that, although the quantities permittivity $\kappa$ and permeability $\mu$ are usually defined as the ratios of magnitudes of the same kind, the assumption that these quantities are, like specific gravity and specific heat, both dimensionless, leads to a fundamental inconsistency of the dimensions of electrical quantities worked out in two different ways. We have seen also that although the dimensions of the quantities $\kappa$ and $\mu$ in terms of $M$, $L$ and $T$ are each unknown, the dimensions of their product $\kappa\mu$ are $L^{-2}T^2$, so that the dimensions of each can be expressed in terms of the unknown dimensions of the other and of $L$ and $T$. Two complete sets of dimensional formulae can therefore be established, the one in terms of $M$, $L$, $T$ and $\kappa$, and the other in terms of $M$, $L$, $T$ and $\mu$.

Although an additional symbol or a fourth dimensional entity seems essential for expressing the dimensions of electrical magnitude, a suggestion has been put forward by Fitzgerald, whereby this additional symbol can be avoided. This suggestion is that the dimensions of $\kappa$ and $\mu$ are assumed to be the same, so that each has the value $L^{-1}T$ to satisfy the dimensional equation $[\kappa\mu]=L^{-2}T^2$. On this assumption, the magnitudes $q$, or quantity of electricity and $m$, pole strength, will have the same dimensions in terms of $M$, $L$ and $T$. Although however the magnitude $\kappa\mu$ certainly has the dimensions $L^{-2}T^2$, there is no justification whatever for assuming that the dimensions of $\kappa$ and $\mu$ are the same. The suggestion that they are is merely a guess, and as the two magnitudes are of a very different physical character, the suggestion seems, on *a priori* grounds, to be inherently improbable.

But, although it has been necessary to assign some unknown dimensions to the quantities $\kappa$ and $\mu$ in the

fundamental laws on which the two systems of electrical units are based, there is no essential reason why either of these quantities need be used as a fourth dimensional entity. We have explained in Chapter II that the choice of fundamental entities can be arbitrary, and that it can be made on the grounds of convenience. The reasons that mass, length and time are used as the fundamental dimensional entities in dynamics are, first, that these entities correspond to concretely and exactly defined units, and, secondly, that the dimensional formulae based upon them are the simplest in the long run. The fourth dimensional entity necessary for expressing the dimensions of electrical quantities can likewise be chosen arbitrarily and this choice can be that of an electrical magnitude measured in a concretely defined unit, or of one which leads to the simplest dimensional formulae. Accordingly, two electrical magnitudes, alternative to $\kappa$ and $\mu$, are used for the fourth dimensional entity; these are resistance, and charge or quantity of electricity.

**Resistance as a Fourth Dimensional Entity.** The use of resistance as a fourth dimensional entity has the sanction that a practical unit of resistance, like a unit of length, can be objectively realised by a column of a liquid metal of a stipulated shape, so that a resistance unit is comparable in simplicity and exactness of definition with a mass and a length unit. The dimensions of electrical quantities in terms of $M$, $L$, $T$ and resistance $R$ can be worked out, starting from Joule's law, that the electrical power loss due to the flow of a current $I$ in a resistance $R$ is $I^2R$. Thus the dimensions of current $[I]$ are given by the relation :

$$[I^2]=\frac{\text{power}}{\text{resistance}}=\frac{ML^2}{T^3}\times\frac{1}{R}=ML^2T^{-3}R^{-1},$$

and $\quad [I]=M^{1/2}LT^{-3/2}R^{-1/2},$

and, as potential difference $V$ is equal to $I\times$ resistance :

$$[V]=M^{1/2}LT^{-3/2}R^{1/2},$$

and, as quantity of electricity $q$ is equal to $I \times t$:

$$[q] = M^{1/2}LT^{-1/2}R^{-1/2}.$$

And so, as we shall show in the next section, the dimensions of all electrical and magnetic quantities can be derived. These other dimensions, in terms of $M$, $L$, $T$ and $R$ are given in the table on page 101. We observe that fractional exponents or indices are necessary in these dimensional formulae.

**Quantity of Electricity as a Fourth Dimensional Entity.** Quantity of electricity is another magnitude which is used as a fourth dimensional entity, and this use can be justified on two grounds. First, that a unit of quantity can be objectively defined in terms of the mass of a metal electrolytically deposited from a specified solution, and secondly, that the dimensional formulae in which quantity is used as a fourth entity are simpler than those in which resistance, permittivity or permeability are so used. The dimensions of electrical magnitudes in terms of $M$, $L$, $T$ and quantity $Q$ are worked out, starting from the principle that quantity × potential difference is equal to electrical energy, so that voltage = energy ÷ quantity. Thus, the dimensions of potential difference $[V]$ are given by:

$$[V] = \frac{ML^2}{T^2} \times \frac{1}{Q} = ML^2T^{-2}Q^{-1}.$$

As current is quantity divided by time:

$$[I] = T^{-1}Q,$$

and as resistance is equal to $V \div I$:

$$[R] = ML^2T^{-2}Q^{-1} \times Q^{-1}T = ML^2T^{-1}Q^{-2}.$$

Resistivity $\rho$ is defined by the equation $R = \frac{\rho l}{a^2}$, where $l$ is the length and $a$ the cross sectional area of a conductor, so that $\rho = \frac{Ra^2}{l}$ and $\rho$ has the dimensions $R \times$ length. Thus:

$$[\rho] = R \times L = ML^3T^{-1}Q^{-2}.$$

Reactance and impedance, each defined as a ratio $V \div I$, have the dimensions of resistance.

Reactance is proportional to inductance × frequency, so that the dimensions of inductance are those of reactance ÷ frequency or of reactance × time. Thus:

$$[\text{Inductance}] = ML^2T^{-1}Q^{-2} \times T = ML^2Q^{-2}.$$

Induced e.m.f. giving a potential difference $V$ is [illegible]

by time Thus:

$$[N] = [V] \times [T] = ML^2T^{-1}Q^{-1},$$

and flux density, $B$, or total flux divided by area has the dimensions given by:

$$[B] = [N] \div L^2 = MT^{-1}Q^{-1}.$$

Magnetising force $H$ in suitable units is equal to current × turns of winding divided by length, so that, as turns of winding is a pure number:

$$[H] = [I] \div L = L^{-1}T^{-1}Q,$$

and as permeability is equal to $B \div H$:

$$[\mu] = MT^{-1}Q^{-1} \times LTQ^{-1} = MLQ^{-2}.$$

Capacitance $C$, being defined by $q \div V$, has the dimensions:

$$[C] = Q \times M^{-1}L^{-2}T^2Q = M^{-1}L^{-2}T^2Q^2.$$

We have seen on page 92 that capacitance has the dimensions $L \times [\kappa]$, where $[\kappa]$ stands for the dimensions of permittivity. Thus:

$$[\kappa] = [C] \div L = M^{-1}L^{-3}T^2Q^2.$$

We see that no fractional indices occur in the dimensional formulae in terms of $M$, $L$, $T$ and $Q$. To those not highly expert in algebraic manipulation, fractional indices introduce a difficulty and a complication in dimensional analysis, and the complete avoidance

**Dimensions of Electrical and Magnetic Quantities**

| *Quantity* | *Fundamental entities* | | | |
|---|---|---|---|---|
| | $M, L, T, \kappa$ | $M, L, T, \mu$ | $M, L, T, R$ | $M, L, T, Q$ |
| Energy | $ML^2T^{-2}$ | $ML^2T^{-2}$ | $ML^2T^{-2}$ | $ML^2T^{-2}$ |
| Power | $ML^2T^{-3}$ | $ML^2T^{-3}$ | $ML^2T^{-3}$ | $ML^2T^{-3}$ |
| Quantity of electricity | $M^{1/2}L^{3/2}T^{-1}\kappa^{1/2}$ | $M^{1/2}L^{1/2}\mu^{-1/2}$ | $M^{1/2}LT^{-1/2}R^{-1/2}$ | $Q$ |
| Current | $M^{1/2}L^{3/2}T^{-2}\kappa^{1/2}$ | $M^{1/2}L^{1/2}T^{-1}\mu^{-1/2}$ | $M^{1/2}LT^{-3/2}R^{-1/2}$ | $T^{-1}Q$ |
| Potential difference | $M^{1/2}L^{1/2}T^{-1}\kappa^{-1/2}$ | $M^{1/2}L^{3/2}T^{-2}\mu^{1/2}$ | $M^{1/2}LT^{-3/2}R^{1/2}$ | $ML^2T^{-2}Q^{-1}$ |
| Resistance, Reactance, Impedance | $L^{-1}T\kappa^{-1}$ | $LT^{-1}\mu$ | $R$ | $ML^2T^{-1}Q^{-2}$ |
| Inductance | $L^{-1}T^2\kappa^{-1}$ | $L\mu$ | $TR$ | $ML^2Q^{-2}$ |
| Total magnetic flux | $M^{1/2}L^{1/2}\kappa^{-1/2}$ | $M^{1/2}L^{3/2}T^{-1}\mu^{1/2}$ | $M^{1/2}LT^{-1/2}R^{1/2}$ | $ML^2T^{-1}Q^{-1}$ |
| Flux density | $M^{1/2}L^{-3/2}\kappa^{-1/2}$ | $M^{1/2}L^{-1/2}T^{-1}\mu^{1/2}$ | $M^{1/2}L^{-1}T^{-1/2}R^{1/2}$ | $MT^{-1}Q^{-1}$ |
| Magnetising force | $M^{1/2}L^{1/2}T^{-2}\kappa^{1/2}$ | $M^{1/2}L^{-1/2}T^{-1}\mu^{-1/2}$ | $M^{1/2}T^{-3/2}R^{1/2}$ | $L^{-1}T^{-1}Q$ |
| Capacitance | $L\kappa$ | $L^{-1}T^2\mu^{-1}$ | $TR^{-1}$ | $M^{-1}L^{-2}T^2Q^2$ |
| Resistivity | $T\kappa^{-1}$ | $L^2T^{-1}\mu$ | $RL$ | $ML^3T^{-1}Q^{-2}$ |
| Permeability | $L^{-2}T^2\kappa^{-1}$ | $\mu$ | $L^{-1}TR$ | $MLQ^{-2}$ |
| Permittivity | $\kappa$ | $L^{-2}T^2\mu^{-1}$ | $L^{-1}TR^{-1}$ | $M^{-1}L^{-3}T^2Q^2$ |

The dimensional equation is, then:

$$ML^2T^{-2}=[ML^2T^{-3}]^x \times [ML^2Q^{-2}]^y \times [ML^2T^{-1}Q^{-2}]^z,$$

and, equating dimensions of $M$, $L$, $T$ and $Q$, we obtain:

$$\begin{aligned} 1 &= x+y+z, \\ 2 &= 2x+2y+2z, \\ -2 &= -3x-z, \\ 0 &= -2y-2z. \end{aligned}$$

From these equations, of which it may be noted the first and second are identical, we easily obtain:

$$x=1, \quad z=-1, \quad \text{and} \quad y=1.$$

The required relationship is therefore:

$$\text{Torque}=b \times P \times \frac{N}{R}.$$

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